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Bernoulli–Euler beam model based on a modified couple stress theory

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Abstract
A new model for the bending of a Bernoulli–Euler beam is developed using a modified couple stress theory. A variational formulation based on the principle of minimum total potential energy is employed. The new model contains an internal material length scale parameter and can capture the size effect, unlike the classical Bernoulli–Euler beam model. The former reduces to the latter in the absence of the material length scale parameter. As a direct application of the new model, a cantilever beam problem is solved. It is found that the bending rigidity of the cantilever beam predicted by the newly developed model is larger than that predicted by the classical beam model. The difference between the deflections predicted by the two models is very significant when the beam thickness is small, but is diminishing with the increase of the beam thickness. A comparison shows that the predicted size effect agrees fairly well with that observed experimentally.

1. Introduction

Thin (cantilever) beams have found important applications in micro- and nano-scale measurements such as those in biosensors and atomic force microscopes (e.g., Pereira (2001), Pei *et al* (2004)). In these applications, the beam thickness is typically on the order of microns, and the size effect (i.e., the thinner, the stiffer) is often observed (e.g., Lam *et al* (2003), McFarland and Colton (2005)). Lacking an internal material length scale parameter, classical beam models cannot be used to interpret this microstructure-dependent size effect and, therefore, need to be extended by using higher order (non-local) continuum theories that contain additional material length scale parameters.

The classical couple stress elasticity theory elaborated by Koiter (1964) and others including Toupin (1962), Mindlin and Tiersten (1962) and Mindlin (1963) is a higher order continuum theory that contains four material constants (two classical and two additional) for isotropic elastic materials. This theory has been applied to model the pure bending of a circular cylinder by Anthoine (2000). Beam bending models based on other non-local elasticity theories have also been reported. For example, the higher order model for Bernoulli–Euler beams developed by Papargyri-Beskou *et al* (2003) is based on the gradient elasticity theory with surface energy of Vardoulakis and Sulem (1995), which involves four elastic constants (two classical and two non-classical). This strain gradient beam model has been studied further by Vardoulakis and Giannakopoulos (2006). The non-local Bernoulli–Euler beam model proposed by Peddieson *et al* (2003) using a constitutive equation due to Eringen (1983) also contains two additional material constants. Considering the difficulties in determining the microstructure related length scale parameters (e.g., Yang and Lakes (1982), Lam *et al* (2003)) and the approximate nature of beam theories, it is desirable to have non-local beam models that involve only one additional material length scale parameter.

A modified couple stress theory has recently been proposed by Yang *et al* (2002), in which the couple stress tensor is symmetric and only one internal material length scale parameter is involved, unlike those in the classical couple stress theory mentioned above. A variational formulation of this modified couple stress theory has subsequently been provided by Park and Gao (2006).

The objective of this paper is to develop a new non-local model for Bernoulli–Euler beams using the minimum total potential energy principle and the concepts of the modified couple stress theory of Yang *et al* (2002). The rest of the paper is organized as follows. In section 2, the strain energy density function is constructed by using the modified couple stress theory and the displacement field typical for a Bernoulli–Euler beam. The principle of minimum total potential energy is then used to obtain the governing equation and boundary conditions for the beam. The resulting beam model contains an internal material length scale parameter and can capture the size effect. To illustrate the newly developed model, a cantilever beam problem is solved in section 3, and the differences between the new beam model and the classical Bernoulli–Euler beam theory are quantitatively shown. The predictions are also compared to and verified by the existing experimental data. The paper concludes with a summary in section 4.

2. Formulation

According to the modified couple stress theory of Yang *et al* (2002), the strain energy density is a function of both strain (conjugated with stress) and curvature (conjugated with couple stress). It then follows that the strain energy U in a deformed isotropic linear elastic material occupying region Ω is given by

$$U = \frac{1}{2}\iiint_{\Omega} (\boldsymbol{\sigma} : \boldsymbol{\varepsilon} + \boldsymbol{m} : \boldsymbol{\chi})\,\mathrm{d}v, \qquad (1)$$

where the stress tensor, $\boldsymbol{\sigma}$, strain tensor, $\boldsymbol{\varepsilon}$, deviatoric part of the couple stress tensor, $\boldsymbol{m}$, and symmetric curvature tensor, $\boldsymbol{\chi}$, are, respectively, defined by

$$\boldsymbol{\sigma} = \lambda\,\mathrm{tr}\,(\boldsymbol{\varepsilon})\mathbf{I} + 2\mu\boldsymbol{\varepsilon}, \qquad (2)$$

$$\boldsymbol{\varepsilon} = \tfrac{1}{2}[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathrm{T}}], \qquad (3)$$

$$\boldsymbol{m} = 2l^2\mu\boldsymbol{\chi}, \qquad (4)$$

$$\boldsymbol{\chi} = \tfrac{1}{2}[\nabla \boldsymbol{\theta} + (\nabla \boldsymbol{\theta})^{\mathrm{T}}], \qquad (5)$$

with λ and μ being Lamé's constants, l a material length scale parameter, $\boldsymbol{u}$ the displacement vector and $\boldsymbol{\theta}$ the rotation vector given by

$$\boldsymbol{\theta} = \tfrac{1}{2}\mathrm{curl}\,\boldsymbol{u}. \qquad (6)$$

Clearly, both $\boldsymbol{\sigma}$ and $\boldsymbol{m}$, as respectively defined in equations (2) and (4), are symmetric, with $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}}$ and $\boldsymbol{m} = \boldsymbol{m}^{\mathrm{T}}$ due to the symmetry of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\chi}$ given in equations (3) and (5), respectively. Also, note that the square of the length scale parameter l introduced in equation (4) is the ratio of the modulus of curvature to the modulus of shear, and l is therefore regarded as a material property measuring the effect of couple stress (Mindlin (1963)).

The work done by external forces is

$$W = \iiint_{\Omega} (\boldsymbol{f} \cdot \boldsymbol{u} + \boldsymbol{c} \cdot \boldsymbol{\theta})\,\mathrm{d}v + \iint_{\partial\Omega} (\boldsymbol{t} \cdot \boldsymbol{u} + \boldsymbol{s} \cdot \boldsymbol{\theta})\,\mathrm{d}a, \qquad (7)$$

where $\boldsymbol{f}$, $\boldsymbol{c}$, $\boldsymbol{t}$ and $\boldsymbol{s}$ are, respectively, the body force, body couple, traction and surface couple, and $\partial\Omega$ is the surface of Ω.

Using the coordinate system (x, y, z) shown in figure 1, where x-axis is coincident with the centroidal axis of the undeformed beam, y-axis is the neutral axis and z-axis is the symmetry axis, the displacement components in a Bernoulli–Euler beam can be represented by (e.g., Shames (1985))

$$u = -z\psi(x), \qquad v = 0, \qquad w = w(x), \qquad (8)$$

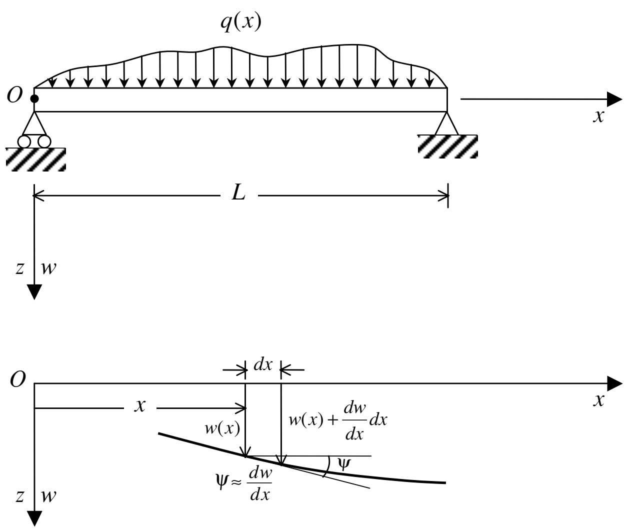

Figure 1. Beam configuration.

where u, v, w are, respectively, the x-, y- and z-components of the displacement vector $\boldsymbol{u}$, and ψ is the rotation angle of the centroidal axis of the beam given approximately by

$$\psi \approx \frac{\mathrm{d}w(x)}{\mathrm{d}x} \qquad (9)$$

for small deformations considered here.

From equations (3), (8) and (9) it follows that

$$\varepsilon_{xx} = -z\frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2}, \qquad \varepsilon_{yy} = \varepsilon_{zz} = \varepsilon_{xy} = \varepsilon_{yz} = \varepsilon_{zx} = 0, \qquad (10)$$

and from equations (6), (8) and (9) that

$$\theta_y = -\frac{\mathrm{d}w(x)}{\mathrm{d}x}, \qquad \theta_x = \theta_z = 0. \qquad (11)$$

Using equation (11) in equation (5) gives

$$\chi_{xy} = -\frac{1}{2}\frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2}, \qquad \chi_{xx} = \chi_{yy} = \chi_{zz} = \chi_{yz} = \chi_{zx} = 0, \qquad (12)$$

and inserting equation (10) into equation (2) yields

$$\sigma_{xx} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left(-z\frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2}\right), \qquad \sigma_{yy} = \sigma_{zz} = \frac{E\nu}{(1+\nu)(1-2\nu)}\left(-z\frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2}\right), \qquad \sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0, \qquad (13)$$

where E, ν are, respectively, Young's modulus and Poisson's ratio of the beam material which are related to Lamé's constants λ and μ by (e.g., Timoshenko and Goodier (1970))

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad (14a)$$

$$\mu = \frac{E}{2(1+\nu)}. \qquad (14b)$$

The material constant μ defined in equation (14*b*) is also known as the shear modulus (often denoted by G). For a slender beam with a large aspect ratio, the Poisson effect is secondary

$(EI + 4\mu A l^2) \frac{d^4 w}{dx^4} = q(x)$

and may be neglected to facilitate the formulation of a simple beam theory (e.g., Shames (1985)). By setting $\nu = 0$, as was done in classical beam theories (e.g., Shames (1985)), equation (13) reduces to

$$\sigma_{xx} = -Ez\frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2}, \qquad \text{all other} \quad \sigma_{ij} = 0. \tag{15}$$

Similarly, the use of equation (12) in equation (4) gives

$$m_{xy} = -\mu l^2 \frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2}, \qquad m_{xx} = m_{yy} = m_{zz} = m_{yz} = m_{zx} = 0, \tag{16}$$

where μ is the shear modulus (see equation (14*b*)).

Substituting equations (10), (12), (15) and (16) into equation (1) then leads to

$$U = -\frac{1}{2}\int_{x=0}^{L} M_x \frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2}\,\mathrm{d}x - \frac{1}{2}\int_{x=0}^{L} Y_{xy}\frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2}\,\mathrm{d}x, \tag{17}$$

where the resultant moment M_x and couple moment Y_{xy} are defined, respectively, by

$$M_x = \int_A \sigma_{xx} z\,\mathrm{d}A, \tag{18a}$$

$$Y_{xy} = \int_A m_{xy}\,\mathrm{d}A. \tag{18b}$$

By neglecting the body force and body couple, the work done by the external forces in the form of transverse loading $q(x)$ shown in figure 1 (without surface couple) is obtained from equation (7) as

$$W = \int_{x=0}^{L} q(x) w(x)\,\mathrm{d}x. \tag{19}$$

From equations (17) and (19) it follows that the total potential energy Π in the loaded beam is

$$\Pi = U - W = -\frac{1}{2}\int_{x=0}^{L}(M_x + Y_{xy})\frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2}\,\mathrm{d}x - \int_{x=0}^{L} q(x) w(x)\,\mathrm{d}x. \tag{20}$$

Taking the first variation of Π gives

$$\delta\Pi = -(M_x + Y_{xy})\delta w'(x)|_0^L + \left(\frac{\mathrm{d}M_x}{\mathrm{d}x} + \frac{\mathrm{d}Y_{xy}}{\mathrm{d}x}\right)\delta w(x)|_0^L - \int_0^L \left(\frac{\mathrm{d}^2 M_x}{\mathrm{d}x^2} + \frac{\mathrm{d}^2 Y_{xy}}{\mathrm{d}x^2} + q\right)\delta w(x)\,\mathrm{d}x. \tag{21}$$

Applying the principle of minimum total potential energy, i.e., $\delta\Pi = 0$ for the stable equilibrium (e.g., Steigmann (1992)), and the fundamental lemma of the calculus of variation (e.g., Gao and Mall (2001)) then leads to, from equation (21),

$$\frac{\mathrm{d}^2 M_x}{\mathrm{d}x^2} + \frac{\mathrm{d}^2 Y_{xy}}{\mathrm{d}x^2} + q(x) = 0, \qquad \forall\, x \in (0, L) \tag{22}$$

as the governing (equilibrium) equation, and

$$\left.\begin{array}{lll} M_x + Y_{xy} & \text{or} & \dfrac{\mathrm{d}w}{\mathrm{d}x} \\ \dfrac{\mathrm{d}(M_x + Y_{xy})}{\mathrm{d}x} & \text{or} & w \end{array}\right\} \quad \text{prescribed at} \quad x = 0 \quad \text{and} \quad x = L \tag{23}$$

as the boundary conditions.

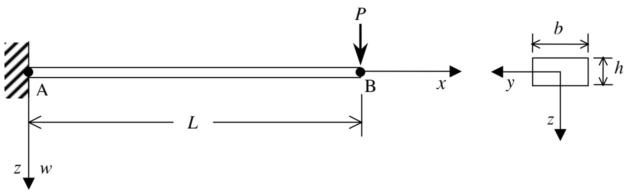

Figure 2. Cantilever beam problem.

From equations (15), (16) and (18*a*, 18*b*) it follows that

$$M_x = -EI\frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2}, \tag{24a}$$

$$Y_{xy} = -\mu A l^2 \frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2}, \tag{24b}$$

where I is the usual second moment of cross-sectional area defined by

$$I = \int_A z^2\,\mathrm{d}A. \tag{25}$$

Substituting equations (24*a*, 24*b*) into equation (22) then gives

$$(EI + \mu A l^2)\frac{\mathrm{d}^4 w(x)}{\mathrm{d}x^4} = q(x) \tag{26}$$

as the equilibrium equation of the beam in terms of $w(x)$. Furthermore, combining equations (24*a*) and (24*b*) yields

$$M_x + Y_{xy} = -(EI + \mu A l^2)\frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2}, \tag{27}$$

which shows that the bending deformation of the beam has two contributions: one associated with the normal stress component σ_{xx} (the conventional term; see equations (18*a*) and (24*a*)) and the other associated with the couple stress component m_{xy} (the additional term; see equations (18*b*) and (24*b*)). Equation (27) also indicates that the bending rigidity of the beam, $(EI + \mu A l^2)$, explicitly depends on l. The value of l is related to and changes with the underlying microstructure of the beam material.

It is seen from equations (22)–(27) that the current beam model based on the modified couple stress theory contains only one additional material constant, i.e., internal material length scale parameter l, unlike the other non-local beam models reviewed in section 1. Nevertheless, the presence of l enables the incorporation of the material microstructural features in the new model and renders it possible to explain the size effect. This will be demonstrated further in the next section.

Clearly, when the microstructural effect is suppressed by letting $l = 0$, the new model defined by equations (22)–(27) will reduce to the classical Bernoulli–Euler beam model.

3. Example: a cantilever beam problem

The Bernoulli–Euler beam model based on the modified couple stress theory of Yang *et al* (2002) is developed in the preceding section. In this section, the problem of a cantilever beam with the loading, geometry and cross-sectional shape shown in figure 2 is solved by directly applying the new model.

Following equation (23), the boundary conditions of this problem are

$$w|_{x=0} = 0, \tag{28a}$$

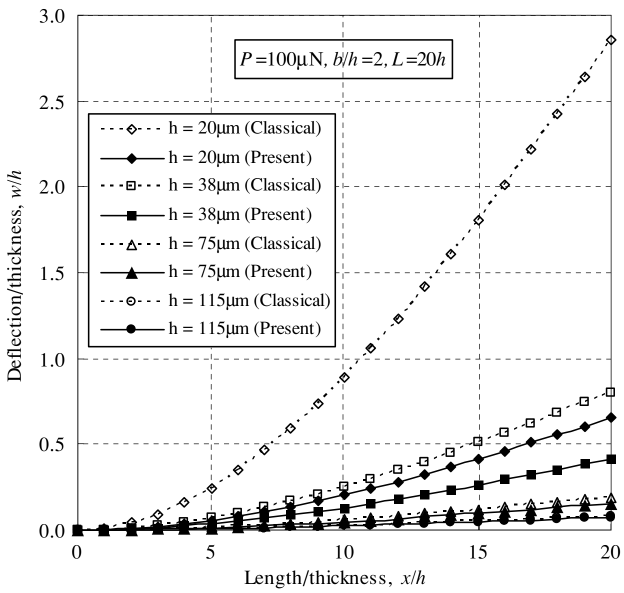

Figure 3. Deflection of the cantilever beam.

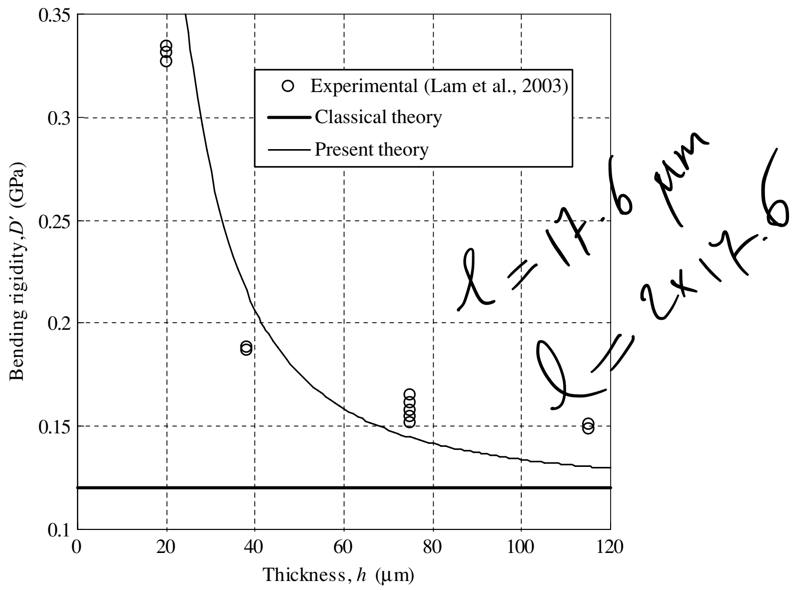

Figure 4. Bending rigidity of the cantilever beam.

$$\left.\frac{\mathrm{d}w}{\mathrm{d}x}\right|_{x=0} = 0, \tag{28b}$$

$$(M_x + Y_{xy})|_{x=L} = 0, \tag{28c}$$

$$\left.\frac{\mathrm{d}(M_x + Y_{xy})}{\mathrm{d}x}\right|_{x=L} = P, \tag{28d}$$

where P is the magnitude of the applied force. Integrating the governing equation given in equation (26) four times gives, with $q(x) = 0$ here,

$$(EI + \mu A l^2)\, w(x) = \frac{C_1}{6}x^3 + \frac{C_2}{2}x^2 + C_3 x + C_4. \tag{29}$$

Using equations (28a)–(28d) in equation (29) will yield, with the help of equations (24a, 24b),

$$C_1 = -P, \qquad C_2 = PL, \qquad C_3 = C_4 = 0. \tag{30}$$

It then follows from equations (29) and (30) that

$$w\,(x) = \frac{Px^2}{6(EI + \mu A l^2)}(3L - x) \tag{31}$$

as the deflection of the beam at the x cross-section. Knowing $w(x)$, all other quantities will be readily determined using the formulae derived in section 2. It should be mentioned that the beam deflection relation given by equation (31) is similar to that provided in McFarland and Colton (2005) for a cantilever plate using a different approach based on the micropolar elasticity theory.

If the microstructural effect, as measured by the internal material length parameter l, is neglected, equation (31) reduces to

$$w\,(x) = \frac{Px^2}{6EI}(3L - x), \tag{32}$$

which is the well-known deflection formula given by the classical Bernoulli–Euler beam theory for the cantilever beam shown in figure 2. A comparison of equations (31) and (32) shows that the classical beam theory predicts a larger deflection than that by the new model based on the modified couple stress theory. This is further illustrated in figure 3.

Figure 3 compares the deflections of the cantilever beam predicted by the new model and by the classical Bernoulli–Euler beam theory. For illustration purpose, the beam considered here is taken to be made of epoxy (Lam *et al* (2003)) with the following properties: $E = 1.44$ GPa, $\nu = 0.38$, $l = 17.6\ \mu$m, with the value of E obtained from figure 9 of Lam *et al* (2003) (for bending tests) and the value of l determined from their equation $(68)_2$ by letting $b_h = 24\ \mu$m, $\nu = 0.38$, and $l_0 = l_1 = 0$, $l_2 \equiv l$ (see Lam *et al* (2003), p 1484, p 1506). Here l_0, l_1 and l_2 are the three material length scale parameters involved in the strain gradient elasticity theory proposed by Lam *et al* (2003) (see their equation (44)), which includes the modified couple stress theory of Yang *et al* (2002) as a special case when the dilatation gradient (measured by l_0) and the deviatoric stretch gradient (measured by l_1) effects are ignored. Also, b_h, called a higher order bending parameter by Lam *et al* (2003), is a material constant related to l_0, l_1, l_2 and ν (Poisson's ratio) (see their equation $(68)_2$). The value of $b_h = 24\ \mu$m used here is taken from Lam *et al* (2003), where it was obtained from curve fitting the experimental data. The cross-sectional shape is kept to be the same by letting $b/h = 2$ (see figure 2) for all cases. The values of P and h have been so chosen that the beam remains elastic everywhere, as was done in Lam *et al* (2003). Clearly, it is seen from figure 3 that the deflection predicted by the classical beam theory is larger than that by the new model along the entire length of the cantilever beam and for all cases considered. Figure 3 also shows that the difference between the two sets of predicted values is very large when the thickness of the beam (h) is on the order of 10 μm but is diminishing when the thickness of the beam becomes larger (h around 100 μm here), thereby indicating that the size effect is only significant at the micron scale. This agrees with the general trends observed in experiments, as shown in figure 4, where the normalized bending rigidity D' predicted by the present beam model is compared to the experimental data provided in Lam *et al* (2003) (see their figure 12). The bending tests of Lam *et al* (2003) were performed using a Hysitron Triboindenter, with the epoxy beam specimens fabricated through casting. The normalized bending rigidity D' for the plane stress beam (with $\nu = 0$) here is given by

$$D' \equiv \frac{EI + \mu A l^2}{bh^3} = \frac{E}{12}\left[1 + 2\left(\frac{b_h}{h}\right)^2\right], \qquad (33)$$

where use has been made of equation (14*b*) and the following relations:

$$I = \frac{bh^3}{12}, \qquad (34a)$$

$$A = bh, \qquad (34b)$$

$$l = \sqrt{\frac{b_h^2}{3(1-\nu)}}, \qquad (34c)$$

with equation (34*c*) obtained from equation $(68)_2$ of Lam *et al* (2003) using $l_0 = l_1 = 0$, $l_2 \equiv l$ for the modified couple stress theory as mentioned above. Figure 4 further illustrates the significant size effect displayed by beams with a small thickness ($h < 100$ μm here): the smaller the thickness (h) is, the larger the normalized bending stiffness (D') is. It is seen from figure 4 that this experimentally observed effect is captured fairly well by the new beam model based on the modified couple stress theory of Yang *et al* (2002). In contrast, the classical beam theory does not have the same capability, as shown in figure 4.

4. Summary

A new model for the bending of a Bernoulli–Euler beam is developed by using the minimum total potential energy principle and a modified couple stress theory. The model contains an internal material length scale parameter to account for the microstructural effect, unlike that in the classical Bernoulli–Euler beam theory. The inclusion of this additional material constant enables the new model to capture the size effect. When the microstructural effect is neglected, the new model reduces to that of the classical beam theory.

A cantilever beam problem is solved by directly applying the newly developed beam model. The solution is compared to that of the classical beam theory for the same problem. The numerical results show that the deflection of the cantilever beam predicted by the new model is always smaller than that by the classical beam model. The smaller the beam thickness, the larger the difference between the deflection values predicted by the two models. However, the difference is diminishing with the increase of the beam thickness. These predictions confirm the size effect at the micron scale observed in bending tests and compare fairly well with the existing experimental data.

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