Composites

Lesson 27

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shape functions

Typically, the dependent unknowns (u) of the problem are approximated using the basic idea that any continuous function can be represented by a linear combination of known functions ϕ_i and undetermined coefficients c_i ($u \approx u_h = \sum c_i \phi_i$). Algebraic relations among the undetermined coefficients c_j are obtained by satisfying the governing equations, in a weighted-integral sense, over each element. The approximation functions ϕ_i are often taken to be polynomials, and they are derived using concepts from interpolation theory. Therefore, they are termed *interpolation functions*. ~~Thus~~

Ritz coefficients

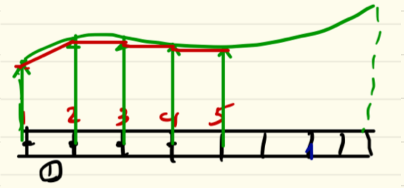
$$u(x) \approx U_N(x) = \sum_{j=1}^{N} C_j \phi_j(x)$$

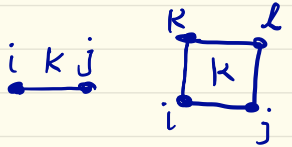
approximate solution

approximation | functions
interpolation

We seek an approximate solution over the entire domain $\Omega = (0,L)$ in the form of above.

We divide (descretize) domain $(0,L)$ into a finite set of N intervals of equal lenth Δx, as shown here. The end of each interval is called a mesh point (nodal point or node). Thus, there are $N+1$ mesh points in the domain.





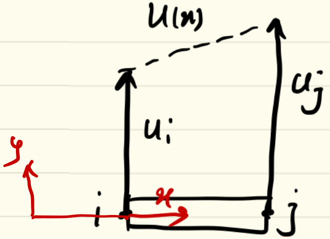
$u(x_1) = u_1$

$u(x_2) = u_2$

$\vdots$

$u(x_i) = u_i$

Over each part, seek an approximation to the solution as a linear combination of nodal values and approximation functions, and derive the algebraic relations among the nodal values of the solution over each part.



$$u = c_0 + c_1 x$$

$$u_i = u(x_i) = c_0 + c_1 (x_i)$$

$$u_j = u(x_j) = c_0 + c_1 (x_j)$$

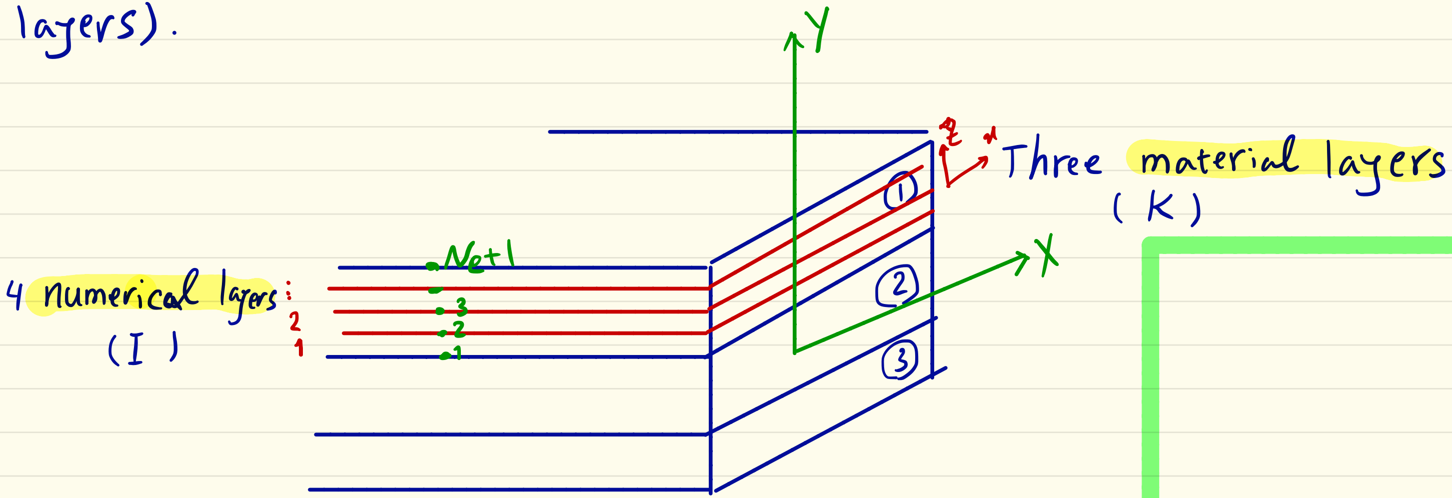
$$\Rightarrow u = u_i \overbrace{\left(\frac{x_j - x}{x_j - x_i}\right)}^{N_i(x)} + u_j \overbrace{\left(\frac{x - x_i}{x_j - x_i}\right)}^{N_j(x)}$$

Instead of representing the solution u as a linear combination ($u_h = \sum_j c_j \phi_j$) in terms of arbitrary parameters c_j as in the variational methods, in the finite element method the solution is often represented as a linear combination ($u_h = \sum_j c_j \phi_j$) in terms of the values u_j of u_h (and possibly its derivatives as well) at the nodal points.

$$\tilde{u} = \sum_{i=1}^{m} u_i N_i(x) = \underbrace{\{N_i(x)\}}_{\text{shape functions}} \underbrace{\{u_i\}}_{\text{nodal values}} \rightarrow \{d\}$$

$$
\begin{cases}
\tilde{u} = c_0 + c_1 x & \left(\equiv c_i \phi_i(x) \right) \\
\tilde{u} = u_i \left(\dfrac{x_j - x}{L_e} \right) + u_j \left(\dfrac{x - x_i}{L_e} \right) & \left(\equiv u_i N_i(x) \right)
\end{cases}
$$

Now we divide each material layer to N_e sublayers (numerical layers).

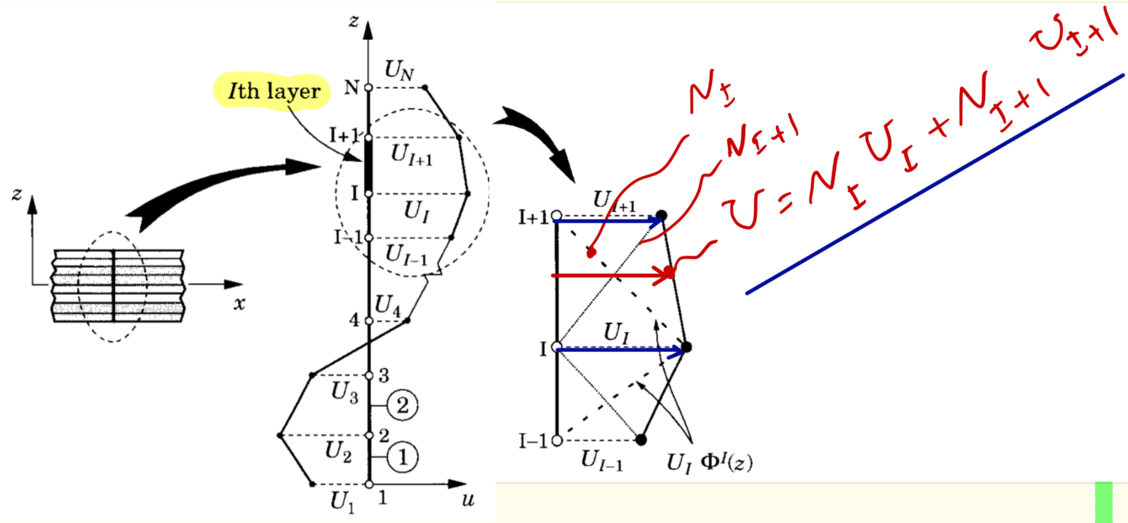


In each material layer we can choose local displacements as:

$$u^k(x,y,z,t) = \sum_{j=1}^{m} u_j^k(x,y,t)\phi_j^k(z)$$
$$v^k(x,y,z,t) = \sum_{j=1}^{m} v_j^k(x,y,t)\phi_j^k(z)$$
$$w^k(x,y,z,t) = \sum_{j=1}^{n} w_j^k(x,y,t)\psi_j^k(z) \quad (9.3\text{-}1)$$

u_j^k, v_j^k, w_j^k: nodal values

ϕ_j^k, ψ_j^k : shape functions (lowercase)



After that, we have to find displacement functions in global coordinates:

The total displacement field of the laminate can be written as

$$u(x,y,z,t) = \sum_{I=1}^{N} U_I(x,y,t)\Phi^I(z)$$
$$v(x,y,z,t) = \sum_{I=1}^{N} V_I(x,y,t)\Phi^I(z)$$
$$w(x,y,z,t) = \sum_{I=1}^{M} W_I(x,y,t)\Psi^I(z)$$

U_I, V_I, W_I : nodal values

Φ, Ψ (uppercase) : global interpolation function (shapefunctions)

(9.3-2)

Now by using Eqs. (9.3-2) (like displacement functions in CLPT and FSDT) we can find other important relations.

$$\begin{cases} u = u_0 + z\dfrac{\partial w}{\partial x} \\ v = v_0 + z + \dfrac{\partial w}{\partial y} \\ w = w_0(x,y) \end{cases}$$

$$\begin{cases} u = u_0 + z\phi_x \\ v = v_0 + z\phi_y \\ w = w_0(x,y) \end{cases}$$

9.3-1 - Strains and Stresses

The von Kármán nonlinear strains associated with the displacement field are

$$
\begin{aligned}
\varepsilon_{xx} &= \sum_{I=1}^{N} \frac{\partial U_I}{\partial x}\Phi^I + \frac{1}{2}\left(\sum_{I=1}^{M} \frac{\partial W_I}{\partial x}\Psi^I\right)\left(\sum_{J=1}^{M} \frac{\partial W_J}{\partial x}\Psi^J\right)\\
\varepsilon_{yy} &= \sum_{I=1}^{N} \frac{\partial V_I}{\partial y}\Phi^I + \frac{1}{2}\left(\sum_{I=1}^{M} \frac{\partial W_I}{\partial y}\Psi^I\right)\left(\sum_{J=1}^{M} \frac{\partial W_J}{\partial y}\Psi^J\right)\\
\gamma_{xy} &= \sum_{I=1}^{N} \left(\frac{\partial U_I}{\partial y} + \frac{\partial V_I}{\partial x}\right)\Phi^I + \left(\sum_{I=1}^{M} \frac{\partial W_I}{\partial x}\Psi^I\right)\left(\sum_{J=1}^{M} \frac{\partial W_J}{\partial y}\Psi^J\right)\\
\varepsilon_{zz} &= \sum_{I=1}^{M} W_I \frac{d\Psi^I}{dz}\\
\gamma_{yz} &= \sum_{I=1}^{N} V_I \frac{d\Phi^I}{dz} + \sum_{I=1}^{M} \frac{\partial W_I}{\partial y}\Psi^I\\
\gamma_{xz} &= \sum_{I=1}^{N} U_I \frac{d\Phi^I}{dz} + \sum_{I=1}^{M} \frac{\partial W_I}{\partial x}\Psi^I
\end{aligned}
$$

(9.3-3)

The stresses in the kth layer may be computed from the 3-D stress-strain equations. For the kth (orthotropic) lamina we have [from Eq. (2.3.19)]

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix}^{(k)} = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13} & 0 & 0 & \bar{C}_{16} \\ \bar{C}_{21} & \bar{C}_{22} & \bar{C}_{23} & 0 & 0 & \bar{C}_{26} \\ \bar{C}_{31} & \bar{C}_{32} & \bar{C}_{33} & 0 & 0 & \bar{C}_{36} \\ 0 & 0 & 0 & \bar{C}_{44} & \bar{C}_{45} & 0 \\ 0 & 0 & 0 & \bar{C}_{45} & \bar{C}_{55} & 0 \\ \bar{C}_{16} & \bar{C}_{26} & \bar{C}_{36} & 0 & 0 & \bar{C}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_{xx} - \alpha_{xx}\Delta T \\ \varepsilon_{yy} - \alpha_{yy}\Delta T \\ \varepsilon_{zz} - \alpha_{zz}\Delta T \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} - 2\alpha_{xy}\Delta T \end{Bmatrix}^{(k)} \quad (9.3\text{-}4)$$

(12.2.8)

9.3-2 Equations of Motion

$$0 = \int_0^T (\delta U + \delta V - \delta K)\, dt$$

(9.3-5)

principle of virtual displacements or works

$$\delta U = \int_{\Omega_0} \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{yy}\delta\varepsilon_{yy} + \sigma_{zz}\delta\varepsilon_{zz} + \right.\right.$$
$$\left.\left. \sigma_{xy}\delta\gamma_{xy} + \sigma_{xz}\delta\gamma_{xz} + \sigma_{yz}\delta\gamma_{yz} \right) dz \right] dxdy$$
$$= \int_{\Omega_0} \left\{ \sum_{I=1}^{N} \left[N_{xx}^I \frac{\partial \delta U_I}{\partial x} + N_{yy}^I \frac{\partial \delta V_I}{\partial y} + N_{xy}^I \left(\frac{\partial \delta U_I}{\partial y} + \frac{\partial \delta V_I}{\partial x} \right) \right.\right.$$
$$\left. + Q_x^I \delta U_I + Q_y^I \delta V_I \right] + \sum_{I=1}^{M} \left[\tilde{Q}_x^I \frac{\partial \delta W_I}{\partial x} + \tilde{Q}_y^I \frac{\partial \delta W_I}{\partial y} \right.$$
$$+ \tilde{Q}_z^I \delta W_I + \left(\tilde{N}_{xx}^{IJ} \frac{\partial W_I}{\partial x} + \tilde{N}_{xy}^{IJ} \frac{\partial W_I}{\partial x} \right) \frac{\partial \delta W_J}{\partial x}$$
$$\left.\left. + \left(\tilde{N}_{xy}^{IJ} \frac{\partial W_I}{\partial x} + \tilde{N}_{yy}^{IJ} \frac{\partial W_I}{\partial y} \right) \frac{\partial \delta W_J}{\partial y} \right] \right\} dxdy$$

(9.3-6a)

$$\delta V = -\int_{\Omega_0}\left[q_b(x,y)\ \delta w(x,y,-\frac{h}{2}) + q_t(x,y)\ \delta w(x,y,\frac{h}{2})\right] dxdy$$
$$-\int_{\Gamma}\int_{-\frac{h}{2}}^{\frac{h}{2}}\left[\hat{\sigma}_{nn}\delta u_n + \hat{\sigma}_{ns}\delta u_s + \hat{\sigma}_{nz}\delta w\right] dzds$$
$$= -\int_{\Omega_0}(q_b\ \delta W_1 + q_t\ \delta W_M)\, dxdy$$
$$-\int_{\Gamma}\left[\sum_{I=1}^{N}\left(\hat{N}^I_{nn}\delta U^n_I + \hat{N}^I_{ns}\delta U^s_I\right) + \sum_{I=1}^{M}\hat{Q}^I_n\delta W_I\right] ds$$

(9.3-6b)

$$\delta K = \int_{\Omega_0}\int_{-\frac{h}{2}}^{\frac{h}{2}}\rho_0\,(\dot{u}\delta\dot{u} + \dot{v}\delta\dot{v} + \dot{w}\delta\dot{w})\, dz\ dxdy$$
$$= \int_{\Omega_0}\left[\sum_{I,J=1}^{N} I^{IJ}\left(\dot{U}_I\delta\dot{U}_J + \dot{V}_I\delta\dot{V}_J\right) + \sum_{I,J=1}^{M}\tilde{I}^{IJ}\dot{W}_I\delta\dot{W}_J\right] dxdy$$

(9.3-6c)

$$\mathcal{N}^I = \sum_{J=1}^{M}\left[\frac{\partial}{\partial x}\left(\tilde{N}^{IJ}_{xx}\frac{\partial W_J}{\partial x} + \tilde{N}^{IJ}_{xy}\frac{\partial W_J}{\partial x}\right) + \frac{\partial}{\partial y}\left(\tilde{N}^{IJ}_{xy}\frac{\partial W_J}{\partial x} + \tilde{N}^{IJ}_{yy}\frac{\partial W_J}{\partial y}\right)\right]$$

(9.3-7)

$$\left\{\begin{matrix} N^I_{xx} \\ N^I_{yy} \\ N^I_{xy}\end{matrix}\right\} = \int_{-\frac{h}{2}}^{\frac{h}{2}}\left\{\begin{matrix}\sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy}\end{matrix}\right\}\Phi^I\, dz, \quad \left\{\begin{matrix} \tilde{N}^{IJ}_{xx} \\ \tilde{N}^{IJ}_{yy} \\ \tilde{N}^{IJ}_{xy}\end{matrix}\right\} = \int_{-\frac{h}{2}}^{\frac{h}{2}}\left\{\begin{matrix}\sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy}\end{matrix}\right\}\Psi^I\Psi^J\, dz$$

(9.3-8)

$$\begin{Bmatrix} Q_x^I \\ Q_y^I \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} \frac{d\Phi^I}{dz}\, dz, \quad \begin{Bmatrix} \tilde{Q}_x^I \\ \tilde{Q}_y^I \\ \tilde{Q}_z^I \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \\ \sigma_{zz} \end{Bmatrix} \Psi^I\, dz \qquad (9.3-9)$$

$$I^{IJ} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho_0 \Phi^I \Phi^J\, dz, \quad \tilde{I}^{IJ} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho_0 \Psi^I \Psi^J\, dz \qquad (9.3-10)$$

$\Rightarrow$

$$\begin{aligned}
\delta U_I: &\quad \frac{\partial N_{xx}^I}{\partial x} + \frac{\partial N_{xy}^I}{\partial y} - Q_x^I = \sum_{J=1}^{N} I^{IJ} \frac{\partial^2 U_J}{\partial t^2} \\
\delta V_I: &\quad \frac{\partial N_{xy}^I}{\partial x} + \frac{\partial N_{yy}^I}{\partial y} - Q_y^I = \sum_{J=1}^{N} I^{IJ} \frac{\partial^2 V_J}{\partial t^2} \\
\delta W_I: &\quad \frac{\partial \tilde{Q}_x^I}{\partial x} + \frac{\partial \tilde{Q}_y^I}{\partial y} - \tilde{Q}_z^I + \tilde{\mathcal{N}}^I + q_b \delta_{I1} + q_t \delta_{IM} = \sum_{J=1}^{M} \tilde{I}^{IJ} \frac{\partial^2 W_J}{\partial t^2}
\end{aligned} \qquad (9.3-11)$$

where

$$N_{nn}^I = N_{xx}^I n_x + N_{xy}^I n_y, \quad N_{ns}^I = N_{xy}^I n_x + N_{yy}^I n_y$$

$$\tilde{Q}_n^I = \tilde{Q}_x^I n_x + \tilde{Q}_y^I n_y, \quad U_I^n = U_I n_x + V_I n_y, \quad U_I^s = -U_I n_x + V_I n_y \qquad (9.3-12)$$

$$\tilde{\mathcal{P}}^I = \sum_{J=1}^{M} \left[\left(\tilde{N}_{xx}^{IJ} \frac{\partial W_J}{\partial x} + \tilde{N}_{xy}^{IJ} \frac{\partial W_J}{\partial x} \right) n_x + \left(\tilde{N}_{xy}^{IJ} \frac{\partial W_J}{\partial x} + \tilde{N}_{yy}^{IJ} \frac{\partial W_J}{\partial y} \right) n_y \right]$$

Primary Variables: $U_I^n, \; U_I^s; \; W_I$

Secondary Variables: $N_{nn}^I, \; N_{ns}^I; \; \tilde{Q}_n^I + \tilde{\mathcal{P}}^I$ (9.3-13)

9.3-3 Laminate Constitutive Equations

$$
\begin{Bmatrix} N_{xx}^I \\ N_{yy}^I \\ N_{xy}^I \end{Bmatrix} = \sum_{k=1}^{N_e} \int_{z_b^k}^{z_t^k} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}^{(k)} \Phi^I \, dz
$$

$$
= \sum_{k=1}^{N_e} \int_{z_b^k}^{z_t^k} \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13} & \bar{C}_{16} \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{23} & \bar{C}_{26} \\ \bar{C}_{16} & \bar{C}_{26} & \bar{C}_{36} & \bar{C}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_{xx} - \alpha_{xx}\Delta T \\ \varepsilon_{yy} - \alpha_{yy}\Delta T \\ \varepsilon_{zz} - \alpha_{zz}\Delta T \\ \gamma_{xy} - 2\alpha_{xy}\Delta T \end{Bmatrix}^{(k)} \Phi^I \, dz \qquad (9.3\text{-}14)
$$

$$
= \sum_{J=1}^{N} \begin{bmatrix} A_{11}^{IJ} & A_{12}^{IJ} & \tilde{A}_{13}^{IJ} & A_{16}^{IJ} \\ A_{12}^{IJ} & A_{22}^{IJ} & \tilde{A}_{23}^{IJ} & A_{26}^{IJ} \\ A_{16}^{IJ} & A_{26}^{IJ} & \tilde{A}_{36}^{IJ} & A_{66}^{IJ} \end{bmatrix} \begin{Bmatrix} \frac{\partial U_J}{\partial x} \\ \frac{\partial V_J}{\partial y} \\ W_J \\ \frac{\partial U_J}{\partial y} + \frac{\partial V_J}{\partial x} \end{Bmatrix} - \begin{Bmatrix} N_{xx}^{I(T)} \\ N_{yy}^{I(T)} \\ N_{xy}^{I(T)} \end{Bmatrix}
$$

$$
+ \sum_{J,K=1}^{M} \begin{bmatrix} B_{11}^{IJK} & B_{12}^{IJK} & B_{16}^{IJK} \\ B_{12}^{IJK} & B_{22}^{IJK} & B_{26}^{IJK} \\ B_{16}^{IJK} & B_{26}^{IJK} & B_{66}^{IJK} \end{bmatrix} \begin{Bmatrix} \frac{1}{2}\frac{\partial W_J}{\partial x}\frac{\partial W_K}{\partial x} \\ \frac{1}{2}\frac{\partial W_J}{\partial y}\frac{\partial W_K}{\partial y} \\ \frac{\partial W_J}{\partial x}\frac{\partial W_K}{\partial y} \end{Bmatrix}
$$

$$
\begin{Bmatrix} \tilde{N}_{xx}^{IJ} \\ \tilde{N}_{yy}^{IJ} \\ \tilde{N}_{xy}^{IJ} \end{Bmatrix} = \sum_{k=1}^{N_e} \int_{z_b^k}^{z_t^k} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}^{(k)} \Psi^I \Psi^J \, dz
$$

$$
= \sum_{K=1}^{N} \begin{bmatrix} B_{11}^{KIJ} & B_{12}^{KIJ} & \tilde{B}_{13}^{KIJ} & B_{16}^{KIJ} \\ B_{12}^{KIJ} & B_{22}^{KIJ} & \tilde{B}_{23}^{KIJ} & B_{26}^{KIJ} \\ B_{16}^{KIJ} & B_{26}^{KIJ} & \tilde{B}_{36}^{KIJ} & B_{66}^{KIJ} \end{bmatrix} \begin{Bmatrix} \frac{\partial U_K}{\partial x} \\ \frac{\partial V_K}{\partial y} \\ W_K \\ \frac{\partial U_K}{\partial y} + \frac{\partial V_K}{\partial x} \end{Bmatrix} - \begin{Bmatrix} \tilde{N}_{xx}^{IJ(T)} \\ \tilde{N}_{yy}^{IJ(T)} \\ \tilde{N}_{xy}^{IJ(T)} \end{Bmatrix} \qquad (9.3\text{-}15)
$$

$$
+ \sum_{K,P=1}^{M} \begin{bmatrix} D_{11}^{IJKP} & D_{12}^{IJKP} & D_{16}^{IJKP} \\ D_{12}^{IJKP} & D_{22}^{IJKP} & D_{26}^{IJKP} \\ D_{13}^{IJKP} & D_{23}^{IJKP} & D_{66}^{IJKP} \end{bmatrix} \begin{Bmatrix} \frac{1}{2}\frac{\partial W_K}{\partial x}\frac{\partial W_P}{\partial x} \\ \frac{1}{2}\frac{\partial W_K}{\partial y}\frac{\partial W_P}{\partial y} \\ \frac{\partial W_K}{\partial x}\frac{\partial W_P}{\partial y} \end{Bmatrix}
$$

$$\left\{\begin{matrix} Q_x^I \\ Q_y^I \end{matrix}\right\} = \sum_{k=1}^{N_e} \int_{z_b^k}^{z_t^k} \left\{\begin{matrix} \sigma_{xz} \\ \sigma_{yz} \end{matrix}\right\}^{(k)} \frac{d\Phi^I}{dz}\, dz = \sum_{k=1}^{N_e} \int_{z_b^k}^{z_t^k} \begin{bmatrix} \bar{C}_{55} & \bar{C}_{45} \\ \bar{C}_{45} & \bar{C}_{44} \end{bmatrix}^{(k)} \left\{\begin{matrix} \gamma_{xz} \\ \gamma_{yz} \end{matrix}\right\}^{(k)} \frac{d\Phi^I}{dz} dz$$
$$= \sum_{J=1}^{N} \left(\begin{bmatrix} \bar{A}_{55}^{IJ} & \bar{A}_{45}^{IJ} \\ \bar{A}_{45}^{IJ} & \bar{A}_{44}^{IJ} \end{bmatrix} \left\{\begin{matrix} U_J \\ V_J \end{matrix}\right\} + \begin{bmatrix} \bar{B}_{55}^{IJ} & \bar{B}_{45}^{IJ} \\ \bar{B}_{45}^{IJ} & \bar{B}_{44}^{IJ} \end{bmatrix} \left\{\begin{matrix} \frac{\partial W_J}{\partial x} \\ \frac{\partial W_J}{\partial y} \end{matrix}\right\} \right) \quad (9.3\text{-}16)$$

$$\left\{\begin{matrix} \tilde{Q}_x^I \\ \tilde{Q}_y^I \end{matrix}\right\} = \sum_{k=1}^{N_e} \int_{z_b^k}^{z_t^k} \left\{\begin{matrix} \sigma_{xz} \\ \sigma_{yz} \end{matrix}\right\}^{(k)} \Psi^I\, dz = \sum_{k=1}^{N_e} \int_{z_b^k}^{z_t^k} \begin{bmatrix} \bar{C}_{55} & \bar{C}_{45} \\ \bar{C}_{45} & \bar{C}_{44} \end{bmatrix}^{(k)} \left\{\begin{matrix} \gamma_{xz} \\ \gamma_{yz} \end{matrix}\right\}^{(k)} \Psi^I dz$$
$$= \sum_{J=1}^{N} \left(\begin{bmatrix} \bar{B}_{55}^{JI} & \bar{B}_{45}^{JI} \\ \bar{B}_{45}^{JI} & \bar{B}_{44}^{JI} \end{bmatrix} \left\{\begin{matrix} U_J \\ V_J \end{matrix}\right\} + \begin{bmatrix} \bar{D}_{55}^{IJ} & \bar{D}_{45}^{IJ} \\ \bar{D}_{45}^{IJ} & \bar{D}_{44}^{IJ} \end{bmatrix} \left\{\begin{matrix} \frac{\partial W_J}{\partial x} \\ \frac{\partial W_J}{\partial y} \end{matrix}\right\} \right) \quad (9.3\text{-}17)$$

where

$$\tilde{Q}_z^I = \sum_{k=1}^{N_e} \int_{z_b^k}^{z_t^k} \sigma_{zz}^{(k)} \frac{d\Psi^I}{dz}\, dz = \sum_{k=1}^{N_e} \int_{z_b^k}^{z_t^k} \left(\bar{C}_{13}\varepsilon_{xx} + \bar{C}_{23}\varepsilon_{yy} + \bar{C}_{33}\varepsilon_{zz} + \bar{C}_{36}\varepsilon_{xy}\right)^{(k)} \frac{d\Psi^I}{dz}\, dz \quad (9.3\text{-}18)$$
$$= \sum_{J=1}^{N} \left[\tilde{A}_{13}^{JI} \frac{\partial U_J}{\partial x} + \tilde{A}_{23}^{JI} \frac{\partial V_J}{\partial y} + \hat{A}_{33}^{JI} W_J + \tilde{A}_{36}^{JI} \left(\frac{\partial U_K}{\partial y} + \frac{\partial V_K}{\partial x} \right) \right] - \tilde{Q}_z^{I(T)}$$