Composites Lesson 26 بسم الله الرحمن الرحيم

8.4-1 Antisymmetric Cross-Ply Laminates (SS-1)

The boundary conditions are satisfied by the following expansions:

$$
\begin{aligned}
u_0(x,y,t) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} U_{mn}(t)\cos\alpha x \sin\beta y \\
v_0(x,y,t) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} V_{mn}(t)\sin\alpha x \cos\beta y \\
w_0(x,y,t) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} W_{mn}(t)\sin\alpha x \sin\beta y \\
\phi_x(x,y,t) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} X_{mn}(t)\cos\alpha x \sin\beta y \\
\phi_y(x,y,t) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} Y_{mn}(t)\sin\alpha x \cos\beta y
\end{aligned}
\qquad (8.4\text{-}3)
$$

where

$$\alpha = \frac{m\pi}{a} \quad \text{and} \quad \beta = \frac{n\pi}{b}$$

$$q(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} Q_{mn}(t) \sin \alpha x \sin \beta y \tag{8.4-4}$$

$$Q_{mn}(t) = \frac{4}{ab} \int_0^a \int_0^b q(x,y,t) \sin \alpha x \sin \beta y \, dx dy \tag{8.4-5}$$

Substituting the expansions into the equations of motion of TSDT plates (8.2-13) we obtain:

$$[\hat{S}]\{\Delta\} + [\hat{M}]\{\ddot{\Delta}\} = \{F\} \tag{8.4-6}$$

mass

where

$$\{\Delta\} = \begin{Bmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \\ X_{mn} \\ Y_{mn} \end{Bmatrix}, \quad \{F\} = \begin{Bmatrix} 0 \\ 0 \\ Q_{mn} \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} \alpha N_{mn}^1 \\ \beta N_{mn}^2 \\ 0 \\ \alpha \hat{M}_{mn}^1 \\ \beta \hat{M}_{mn}^2 \end{Bmatrix} \tag{8.4-7}$$

where $\hat{s}_{ij}$ and $\hat{m}_{ij}$ are defined by:

$$
\begin{aligned}
&\hat{s}_{11} = A_{11}\alpha^2 + A_{66}\beta^2, \quad \hat{s}_{12} = (A_{12} + A_{66})\alpha\beta \\
&\hat{s}_{13} = -c_1\left[E_{11}\alpha^2 + (E_{12} + 2E_{66})\beta^2\right]\alpha \\
&\hat{s}_{14} = \hat{B}_{11}\alpha^2 + \hat{B}_{66}\beta^2, \quad \hat{s}_{15} = (\hat{B}_{12} + \hat{B}_{66})\alpha\beta \\
&\hat{s}_{22} = A_{66}\alpha^2 + A_{22}\beta^2, \quad \hat{s}_{24} = \hat{s}_{15} \\
&\hat{s}_{23} = -c_1\left[E_{22}\beta^2 + (E_{12} + 2E_{66})\alpha^2\right]\beta, \quad \hat{s}_{25} = \hat{B}_{66}\alpha^2 + \hat{B}_{22}\beta^2 \\
&\hat{s}_{33} = \bar{A}_{55}\alpha^2 + \bar{A}_{44}\beta^2 + c_1^2\left[H_{11}\alpha^4 + 2(H_{12} + 2H_{66})\alpha^2\beta^2 + H_{22}\beta^4\right]
\end{aligned}
$$

(8.4-8)

$$
\begin{aligned}
&\hat{s}_{34} = \bar{A}_{55}\alpha - c_1\left[\hat{F}_{11}\alpha^3 + (\hat{F}_{12} + 2\hat{F}_{66})\alpha\beta^2\right] \\
&\hat{s}_{35} = \bar{A}_{44}\beta - c_1\left[\hat{F}_{22}\beta^3 + (\hat{F}_{12} + 2\hat{F}_{66})\alpha^2\beta\right] \\
&\hat{s}_{44} = \bar{A}_{55} + \bar{D}_{11}\alpha^2 + \bar{D}_{66}\beta^2, \quad \hat{s}_{45} = (\bar{D}_{12} + \bar{D}_{66})\alpha\beta \\
&\hat{s}_{55} = \bar{A}_{44} + \bar{D}_{66}\alpha^2 + \bar{D}_{22}\beta^2, \quad \tilde{s}_{33} = \hat{N}_{xx}\alpha^2 + \hat{N}_{yy}\beta^2 \\
&\hat{m}_{11} = I_0, \quad \hat{m}_{22} = I_0, \quad \hat{m}_{33} = I_0 + c_1^2 I_6\left(\alpha^2 + \beta^2\right), \quad \hat{m}_{34} = -c_1 J_4\alpha \\
&\hat{m}_{35} = -c_1 J_4\beta, \quad \hat{m}_{44} = K_2, \quad \hat{m}_{55} = K_2 \\
&\hat{A}_{ij} = A_{ij} - c_1 D_{ij}, \quad \hat{B}_{ij} = B_{ij} - c_1 E_{ij}, \quad \hat{D}_{ij} = D_{ij} - c_1 F_{ij} \ (i,j = 1,2,6) \\
&\hat{F}_{ij} = F_{ij} - c_1 H_{ij}, \quad \bar{A}_{ij} = \hat{A}_{ij} - c_1\hat{D}_{ij} = A_{ij} - 2c_1 D_{ij} + c_1^2 F_{ij} \ (i,j = 1,2,6) \\
&\bar{D}_{ij} = \hat{D}_{ij} - c_1\hat{F}_{ij} = D_{ij} - 2c_1 F_{ij} + c_1^2 H_{ij} \ (i,j = 1,2,6) \\
&\bar{A}_{ij} = \hat{A}_{ij} - c_2\hat{D}_{ij} = A_{ij} - 2c_2 D_{ij} + c_2^2 F_{ij} \ (i,j = 4,5) \\
&J_i = I_i - c_1 I_{i+2}, \quad K_2 = I_2 - 2c_1 I_4 + c_1^2 I_6, \quad c_1 = \frac{4}{3h^2}, \quad c_2 = 3c_1
\end{aligned}
$$

The termal resultants are defined by

$$\begin{Bmatrix} N_{xx}^T \\ N_{yy}^T \\ N_{xy}^T \end{Bmatrix} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \begin{Bmatrix} N_{mn}^1(t) \\ N_{mn}^2(t) \\ N_{mn}^6(t) \end{Bmatrix} \sin\alpha x \, \sin\beta y$$

$$\begin{Bmatrix} M_{xx}^T \\ M_{yy}^T \\ M_{xy}^T \end{Bmatrix} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \begin{Bmatrix} M_{mn}^1(t) \\ M_{mn}^2(t) \\ M_{mn}^6(t) \end{Bmatrix} \sin\alpha x \, \sin\beta y \qquad (8.4\text{-}9)$$

$$\begin{Bmatrix} P_{xx}^T \\ P_{yy}^T \\ P_{xy}^T \end{Bmatrix} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \begin{Bmatrix} P_{mn}^1(t) \\ P_{mn}^2(t) \\ P_{mn}^6(t) \end{Bmatrix} \sin\alpha x \, \sin\beta y$$

$$\{N_{mn}^T(t)\} = \sum_{k=1}^{N}\int_{z_k}^{z_{k+1}} [\bar{Q}]^{(k)}\{\bar{\alpha}\}^{(k)}\, T_{mn}(z,t)\, dz$$

$$\{M_{mn}^T(t)\} = \sum_{k=1}^{N}\int_{z_k}^{z_{k+1}} [\bar{Q}]^{(k)}\{\bar{\alpha}\}^{(k)}\, T_{mn}(z,t)\, z\, dz$$

$$\{P_{mn}^T(t)\} = \sum_{k=1}^{N}\int_{z_k}^{z_{k+1}} [\bar{Q}]^{(k)}\{\bar{\alpha}\}^{(k)}\, T_{mn}(z,t)\, z^3\, dz \qquad (8.4\text{-}10)$$

$$\Delta T(x,y,z,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(T_{mn}^0 + zT_{mn}^1\right)\sin\alpha x \, \sin\beta y$$

$$\left(T_{mn}^0, T_{mn}^1\right) = \frac{4}{ab}\int_0^a\int_0^b (T_0, T_1)\sin\alpha x \, \sin\beta y \, dxdy$$

$$\{\hat{M}_{mn}\} = \{M_{mn}\} - c_1\{P_{mn}\}$$

The ordinary differential equations (8.4-6) in time can be solved for transient response using the Newmark integration procedure described before.

After finding displacements, the stresses in each layer can be computed by:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}^{(k)} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & 0 \\ \bar{Q}_{12} & \bar{Q}_{22} & 0 \\ 0 & 0 & \bar{Q}_{66} \end{bmatrix}^{(k)} \left(\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} - \begin{Bmatrix} \alpha_{xx} \\ \alpha_{yy} \\ 2\alpha_{xy} \end{Bmatrix}^{(k)} \Delta T \right) \quad (8.4\text{-}11)$$

where

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + z \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} + z^3 \begin{Bmatrix} \varepsilon_{xx}^{(3)} \\ \varepsilon_{yy}^{(3)} \\ \gamma_{xy}^{(3)} \end{Bmatrix}$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \begin{Bmatrix} (R_{mn}^{xx} + zS_{mn}^{xx} + c_1 z^3 T_{mn}^{xx}) \sin \alpha x \, \sin \beta y \\ (R_{mn}^{yy} + zS_{mn}^{yy} + c_1 z^3 T_{mn}^{yy}) \sin \alpha x \, \sin \beta y \\ (R_{mn}^{xy} + zS_{mn}^{xy} + c_1 z^3 T_{mn}^{xy}) \cos \alpha x \cos \, \beta y \end{Bmatrix}$$

(8.4-12)

$$\begin{Bmatrix} R_{mn}^{xx} \\ R_{mn}^{yy} \\ R_{mn}^{xy} \end{Bmatrix} = \begin{Bmatrix} -\alpha U_{mn} \\ -\beta V_{mn} \\ \beta U_{mn} + \alpha V_{mn} \end{Bmatrix}, \quad \begin{Bmatrix} S_{mn}^{xx} \\ S_{mn}^{yy} \\ S_{mn}^{xy} \end{Bmatrix} = \begin{Bmatrix} -\alpha X_{mn} \\ -\beta Y_{mn} \\ \beta X_{mn} + \alpha Y_{mn} \end{Bmatrix}$$

$$\begin{Bmatrix} T_{mn}^{xx} \\ T_{mn}^{yy} \\ T_{mn}^{xy} \end{Bmatrix} = \begin{Bmatrix} \alpha X_{mn} + \alpha^2 W_{mn} \\ \beta Y_{mn} + \beta^2 W_{mn} \\ -(\beta X_{mn} + \alpha Y_{mn} + 2\alpha\beta W_{mn}) \end{Bmatrix}$$

The transverse shear stresses from the constitutive equations are given by:

$$\begin{Bmatrix} \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix}^{(k)} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \begin{bmatrix} \bar{Q}_{44} & 0 \\ 0 & \bar{Q}_{55} \end{bmatrix}^{(k)} \left(\begin{Bmatrix} \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \end{Bmatrix} + z^2 \begin{Bmatrix} \gamma_{yz}^{(2)} \\ \gamma_{xz}^{(2)} \end{Bmatrix} \right)$$

$$= (1 - c_2 z^2) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \begin{bmatrix} \bar{Q}_{44} & 0 \\ 0 & \bar{Q}_{55} \end{bmatrix}^{(k)} \begin{Bmatrix} (Y_{mn} + \beta W_{mn}) \sin \alpha x \, \cos \beta y \\ (X_{mn} + \alpha W_{mn}) \cos \alpha x \, \sin \beta y \end{Bmatrix}$$

(8.4-13)

where $c_2 = \frac{4}{h^2}$

8.4.2 Antisymmetric Angle-Ply Laminates (SS-2)

The simply supported (SS-2) boundary conditions are satisfied by:

$$u_0(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} U_{mn}(t)\sin\alpha x\cos\beta y$$
$$v_0(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} V_{mn}(t)\cos\alpha x\sin\beta y \quad (8.4\text{-}14)$$

and (w_0, ϕ_x, ϕ_y) have the same expansions as in Eqs. (8.4-3).

Substituting the expansions in the governing equations (8.2-13) we obtain equations of the form:

$$[\hat{S}]\{\Delta\} + [\hat{M}]\{\ddot{\Delta}\} = \{F\} \quad (8.4\text{-}15)$$

where

$$
\begin{aligned}
&\hat{s}_{11} = A_{11}\alpha^2 + A_{66}\beta^2, \quad \hat{s}_{12} = (A_{12} + A_{66})\alpha\beta \\
&\hat{s}_{13} = -c_1\left(3E_{16}\alpha^2 + E_{26}\beta^2\right)\beta \\
&\hat{s}_{14} = 2\hat{B}_{16}\alpha\beta, \quad \hat{s}_{15} = \hat{B}_{16}\alpha^2 + \hat{B}_{26}\beta^2 \\
&\hat{s}_{22} = A_{66}\alpha^2 + A_{22}\beta^2, \quad \hat{s}_{23} = -c_1\left(E_{16}\alpha^2 + 3E_{26}\beta^2\right)\alpha \\
&\hat{s}_{24} = \hat{s}_{15}, \quad \hat{s}_{25} = 2\hat{B}_{26}\alpha\beta \\
&\hat{s}_{33} = \bar{A}_{55}\alpha^2 + \bar{A}_{44}\beta^2 + c_1^2\left[H_{11}\alpha^4 + 2(H_{12} + 2H_{66})\alpha^2\beta^2 + H_{22}\beta^4\right] \\
&\hat{s}_{34} = \bar{A}_{55}\alpha - c_1\left[\hat{F}_{11}\alpha^3 + (\hat{F}_{12} + 2\hat{F}_{66})\alpha\beta^2\right] \\
&\hat{s}_{35} = \bar{A}_{44}\beta - c_1\left[\hat{F}_{22}\beta^3 + (\hat{F}_{12} + 2\hat{F}_{66})\alpha^2\beta\right] \\
&\hat{s}_{44} = \bar{A}_{55} + \bar{D}_{11}\alpha^2 + \bar{D}_{66}\beta^2, \quad \hat{s}_{45} = (\bar{D}_{12} + \bar{D}_{66})\alpha\beta \\
&\hat{s}_{55} = \bar{A}_{44} + \bar{D}_{66}\alpha^2 + \bar{D}_{22}\beta^2
\end{aligned}
$$

(8.4-16)

The mass and coefficients whit hat and overbar are the same as those defined in the previous.

Chapter IX - Layerwise Theory and Variable Kinematic Models

9.1 Introduction

What we have done so far is to consider one set of constitutive equations or governing equations for the whole laminate. Such methods are called Equivalent Single-layer Laminate theory (ESL theories).

The simple ESL laminate theories are most often incapable of accurately determining the 3-D stress field at the ply level.

Thus the analysis of primary composite structural components may require the use of 3-D elasticity theory or a layerwise laminate theory that contains full 3-D kinematics and constitutive relations.

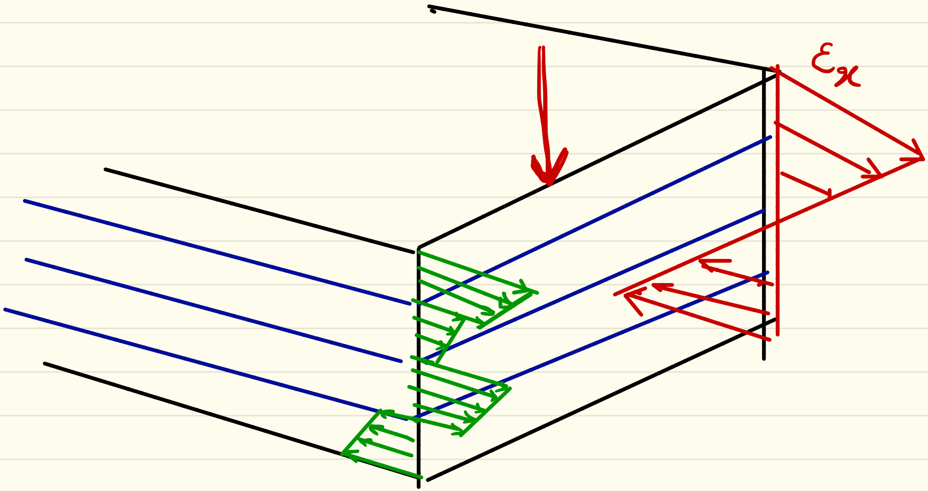
In layerwise method we have equilibrium of interlaminar forces: inplane stresses transvers stresses

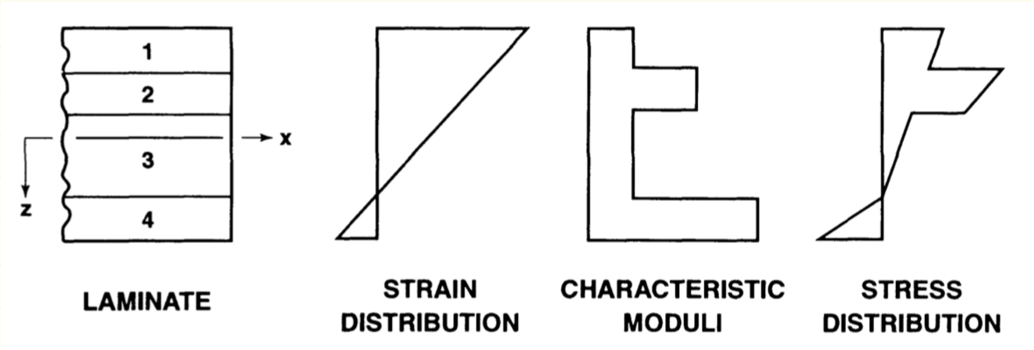
$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}^{(k)} \neq \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}^{(k+1)}, \quad \begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \\ \sigma_{zz} \end{Bmatrix}^{(k)} = \begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \\ \sigma_{zz} \end{Bmatrix}^{(k+1)} \qquad (9.1\text{-}1)$$

Since $[\bar{Q}]^{k} \neq [\bar{Q}]^{k+1}$ in general, the strain fields of adjacent layers do not satisfy continuity conditions:

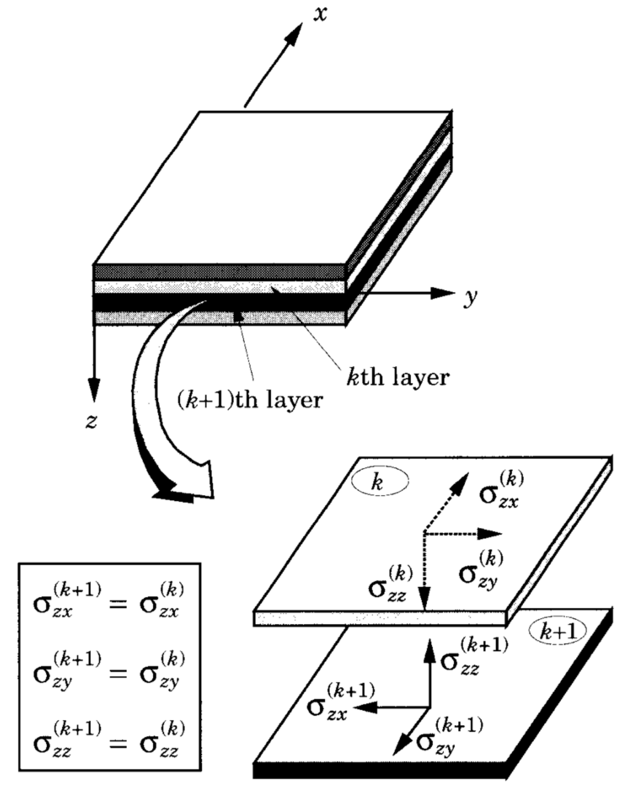
$$\begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \\ \varepsilon_{zz} \end{Bmatrix}^{(k)} \neq \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \\ \varepsilon_{zz} \end{Bmatrix}^{(k+1)} \qquad (9.1\text{-}2)$$

Hence, all stresses in equivalent-single layer theories (ESL) are discontinuous at layer interfaces:


ε_x


1
2
3
4
x
z
LAMINATE
STRAIN DISTRIBUTION
CHARACTERISTIC MODULI
STRESS DISTRIBUTION

$$\begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \\ \sigma_{zz} \end{Bmatrix}^{(k)}_{ESL} \neq \begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \\ \sigma_{zz} \end{Bmatrix}^{(k+1)}_{ESL}$$ in ESL thearies (9.1-3)



9.2- An overview of Layerwise Theories

In contrast of the ESL theories, the layerwise theories are developed by assuming that the displacement field exhibits only C^0-continuity through the laminate thickness. Thus the displacement components are continuous through the laminate thickness but the drivatives of the displacements with respect to the thickness coordinate may be discontinuous at various points through the thickness, separating dissimilar materials.

Layerwise displacement fields provide a much more kinematically correct representation of the moderate to severe cross-sectional warping associated with the deformation of thick laminates.

The displacement-based layerwise theories can be subdivided into two classes:

(1) the partial layerwise theories that use layerwise expansions for the in-plane displacement components but not the transverse displacement component, and

(2) the full layerwise theories that use layerwise expansions for all three displacement components.

More realistic

1- Full Layer wise

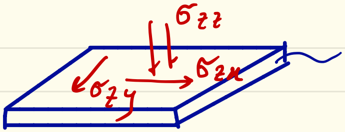
2- partial Layerwise

3- ESL Theories

There are several approaches that have been used to modeling laminates in a layer wise manner:

I- model . 1

This approach involves writing the equilibrium equations for Kth lamina in terms of the force and moment resultants:



layer number K

$$\frac{\partial N_{\alpha\beta}^{(k)}}{\partial x_\beta} + \sigma_{\alpha 3}^{(k+1)} - \sigma_{\alpha 3}^{(k)} = 0, \quad \frac{\partial Q_\alpha^{(k)}}{\partial x_\alpha} + \sigma_{33}^{(k+1)} - \sigma_{33}^{(k)} = 0 \qquad (9.2\text{-}1)$$

$$\frac{\partial M_{\alpha\beta}^{(k)}}{\partial x_\beta} + \sigma_{\alpha 3}^{(k+1)} z_{k+1} - \sigma_{\alpha 3}^{(k)} z_k - Q_\alpha^k = 0$$

II - model 2

In the Zigzag theory or discrete-layer theory the displacement field is assumed to be of the form

$$u_\alpha(x_\beta, x_3, t) = u_\alpha^0(x_\beta, t) - \overset{z}{x_3} u_{3,\alpha}^0(x_\beta, t) + f_{\alpha\gamma}(\overset{z}{x_3})\phi_\gamma(x_\beta, t) \qquad (9.2\text{-}2)$$

$$u_3(x_\beta, x_3, t) = u_3^0(x_\beta, t)$$

$\alpha, \beta, \gamma = 1, 2$

$x_1 = x, \; x_2 = y, \; x_3 = z$

The functions $f_{\alpha\gamma}$ and ϕ_γ are then determind such that the displacements and transverse stresses are continuous.

III model 3

A more direct method of achieving a layerwise displacement field was proposed by Reddy, who repressented the transverse variation of the displacement components in terms of one-dimensional Lagrangian finite elements.

9.3 The Layerwise Theory of Reddy

In the layerwise theory of Reddy, the displacements of the kth layer are written as

$$
\begin{aligned}
u^k(x,y,z,t) &= \sum_{j=1}^{m} u_j^k(x,y,t)\phi_j^k(z) \\
v^k(x,y,z,t) &= \sum_{j=1}^{m} v_j^k(x,y,t)\phi_j^k(z) \\
w^k(x,y,z,t) &= \sum_{j=1}^{n} w_j^k(x,y,t)\psi_j^k(z)
\end{aligned}
\qquad (9.3\text{-}1)
$$

The functions $\phi_j^k(z)$ and $\psi_j^k(z)$ are selected to be layerwise continuous functions. For example, they can be chosen to be the one-dimensional Lagrange interpolation functions of the thickness coordinate, in which case, (u_j^k, v_j^k, w_j^k) denote the values of (u^k, v^k, w^k) at the jth plane (see [37,47–52]).