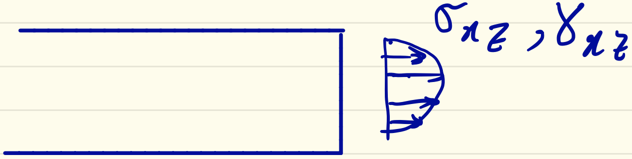
Composites Lesson 25 بسم الله الرحمن الرحيم

chapter VIII Third-order Theory of Laminated Composite Plates

8.1- Introduction

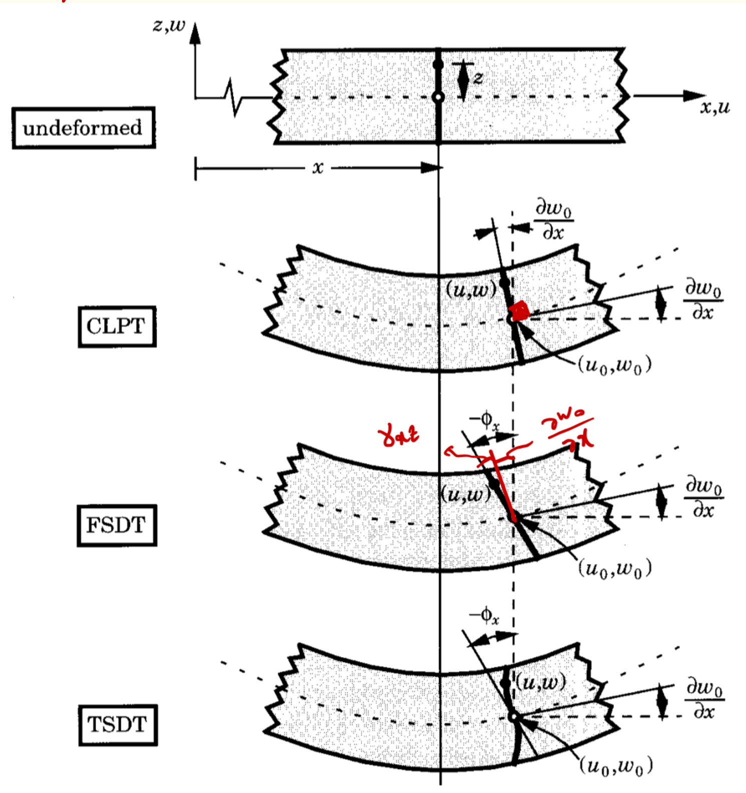
Higher-order theories can represent the Kinematics better, may not require shear correction factors, and can yield more accurate interlaminar stress distributions. However, they involve higher-order stress resultants that are difficult to interpret physically and require considerably more computational effort. Therefore, such theories should be used only when necessary.

The reason for expanding the displacements up to the cubic term in the thickness coordinate is to have quadratic variation of the transverse shear strains and transverse shear stresses through each layer.



8.2 A Third-order Plate Theory

8.2.1 Displacement Field

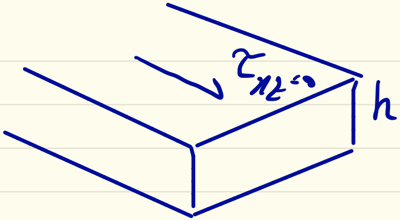


$$u = u_0 + z\phi_x + z^2\theta_x + z^3\lambda_x$$
$$v = v_0 + z\phi_y + z^2\theta_y + z^3\lambda_y \qquad (8.2\text{-}1)$$
$$w = w_0$$

$u_0 = u(x, y, 0, t)$, $v_0 = v(x, y, 0, t)$, $w_0 = w(x, y, 0, t)$

$$\phi_x = \left(\frac{\partial u}{\partial z}\right)_{z=0}, \quad \phi_y = \left(\frac{\partial v}{\partial z}\right)_{z=0}, \quad 2\theta_x = \left(\frac{\partial^2 u}{\partial z^2}\right)_{z=0} \qquad (8.2\text{-}2)$$
$$, 2\theta_y = \left(\frac{\partial^2 v}{\partial z^2}\right)_{z=0}, \quad 6\lambda_x = \left(\frac{\partial^3 u}{\partial z^3}\right)_{z=0}, \quad 6\lambda_y = \left(\frac{\partial^3 v}{\partial z^3}\right)_{z=0}$$

Suppose that we wish to impose traction-free boundary conditions on the top and bottom faces of the laminate.



$$\sigma_{xz}\left(x, y, \pm \frac{h}{2}, t\right) = 0, \quad \sigma_{yz}\left(x, y, \pm h/2, t\right) = 0 \qquad (8.2\text{-}3)$$

$$\Rightarrow \begin{aligned} 0 &= \sigma_{xz}(x, y, \pm h/2, t) = Q_{55}\gamma_{xz}(x, y, \pm h/2, t) + Q_{45}\gamma_{yz}(x, y, \pm h/2, t), \\ 0 &= \sigma_{yz}(x, y, \pm h/2, t) = Q_{45}\gamma_{xz}(x, y, \pm h/2, t) + Q_{44}\gamma_{yz}(x, y, \pm h/2, t) \end{aligned} \qquad (8.2\text{-}4)$$

$$\begin{aligned} \phi_x + \frac{\partial w_0}{\partial x} + \left(-h\theta_x + \frac{3h^2}{4}\lambda_x\right) &= 0, \quad \phi_x + \frac{\partial w_0}{\partial x} + \left(h\theta_x + \frac{3h^2}{4}\lambda_x\right) = 0 \\ \phi_y + \frac{\partial w_0}{\partial y} + \left(-h\theta_y + \frac{3h^2}{4}\lambda_y\right) &= 0, \quad \phi_y + \frac{\partial w_0}{\partial y} + \left(h\theta_y + \frac{3h^2}{4}\lambda_y\right) = 0 \end{aligned}$$

or

$$\lambda_x = -\frac{4}{3h^2}\left(\phi_x + \frac{\partial w_0}{\partial x}\right), \quad \theta_x = 0; \quad \lambda_y = -\frac{4}{3h^2}\left(\phi_y + \frac{\partial w_0}{\partial y}\right), \quad \theta_y = 0$$

$$
\begin{aligned}
u(x,y,z,t) &= u_0(x,y,t) + z\phi_x(x,y,t) - \underbrace{\frac{4}{3h^2}}_{c_1} z^3\left(\phi_x + \frac{\partial w_0}{\partial x}\right)\\
v(x,y,z,t) &= v_0(x,y,t) + z\phi_y(x,y,t) - \frac{4}{3h^2} z^3\left(\phi_y + \frac{\partial w_0}{\partial y}\right)\\
w(x,y,z,t) &= w_0(x,y,t)
\end{aligned}
$$

Displacements in TSDT (8.2-6)

8.2-2 strains

$$
\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + z \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} + z^3 \begin{Bmatrix} \varepsilon_{xx}^{(3)} \\ \varepsilon_{yy}^{(3)} \\ \gamma_{xy}^{(3)} \end{Bmatrix}
$$

$$
\begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{Bmatrix} \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \end{Bmatrix} + z^2 \begin{Bmatrix} \gamma_{yz}^{(2)} \\ \gamma_{xz}^{(2)} \end{Bmatrix}
$$

(8.2-7)

Consider $c_2 = 3c_1$, $c_1 = \dfrac{4}{3h^2}$ then

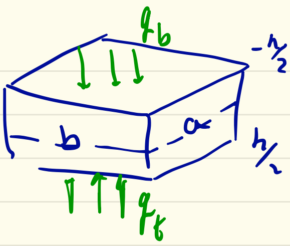
$$\begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 \\ \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2 \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y} \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \end{Bmatrix}, \quad \begin{Bmatrix} \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \end{Bmatrix} = \begin{Bmatrix} \phi_y + \frac{\partial w_0}{\partial y} \\ \phi_x + \frac{\partial w_0}{\partial x} \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_{xx}^{(3)} \\ \varepsilon_{yy}^{(3)} \\ \gamma_{xy}^{(3)} \end{Bmatrix} = -c_1 \begin{Bmatrix} \frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2} \\ \frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w_0}{\partial y^2} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2\frac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix}, \quad \begin{Bmatrix} \gamma_{yz}^{(2)} \\ \gamma_{xz}^{(2)} \end{Bmatrix} = -c_2 \begin{Bmatrix} \phi_y + \frac{\partial w_0}{\partial y} \\ \phi_x + \frac{\partial w_0}{\partial x} \end{Bmatrix}$$

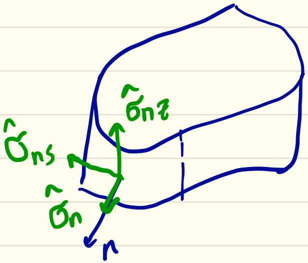
(8.2-8)

8.2-3 Equations of Motion



$$
\begin{aligned}
\delta U = \int_{\Omega_0} \Bigg\{ \int_{-\frac{h}{2}}^{\frac{h}{2}} \Bigg[& \sigma_{xx} \left(\delta\varepsilon_{xx}^{(0)} + z\delta\varepsilon_{xx}^{(1)} - c_1 z^3 \delta\varepsilon_{xx}^{(3)} \right) \\
& + \sigma_{yy} \left(\delta\varepsilon_{yy}^{(0)} + z\delta\varepsilon_{yy}^{(1)} - c_1 z^3 \delta\varepsilon_{yy}^{(3)} \right) + \sigma_{xy} \left(\delta\gamma_{xy}^{(0)} + z\delta\gamma_{xy}^{(1)} - c_1 z^3 \delta\gamma_{xy}^{(3)} \right) \\
& + \sigma_{xz} \left(\delta\gamma_{xz}^{(0)} + z^2 \delta\gamma_{xz}^{(2)} \right) + \sigma_{yz} \left(\delta\gamma_{yz}^{(0)} + z^2 \delta\gamma_{yz}^{(2)} \right) \Bigg] dz \Bigg\} dxdy \\
= \int_{\Omega_0} \Bigg(& N_{xx}\delta\varepsilon_{xx}^{(0)} + M_{xx}\delta\varepsilon_{xx}^{(1)} - c_1 P_{xx}\delta\varepsilon_{xx}^{(3)} + N_{yy}\delta\varepsilon_{yy}^{(0)} + M_{yy}\delta\varepsilon_{yy}^{(1)} - c_1 P_{yy}\delta\varepsilon_{yy}^{(3)} \\
& + N_{xy}\delta\gamma_{xy}^{(0)} + M_{xy}\delta\gamma_{xy}^{(1)} - c_1 P_{xy}\delta\gamma_{xy}^{(3)} \\
& + Q_x\delta\gamma_{xz}^{(0)} - c_2 R_x \delta\gamma_{xz}^{(2)} + Q_y\delta\gamma_{yz}^{(0)} - c_2 R_y \delta\gamma_{yz}^{(2)} \Bigg) dxdy
\end{aligned}
$$

(8.2-9)



$$
\begin{aligned}
\delta V = & - \int_{\Omega_0} \left[q_b(x,y)\delta w(x,y,-\frac{h}{2}) + q_t(x,y)\delta w(x,y,\frac{h}{2}) \right] dxdy \\
& - \int_{\Gamma} \int_{-\frac{h}{2}}^{\frac{h}{2}} \Bigg[\hat{\sigma}_{nn} \left(\delta u_n + z\delta\phi_n - c_1 z^3 \delta\varphi_n \right) \\
& \qquad + \hat{\sigma}_{ns} \left(\delta u_s + z\delta\phi_s - c_1 z^3 \delta\varphi_{ns} \right) + \hat{\sigma}_{nz}\delta w_0 \Bigg] dz d\Gamma \\
= & - \int_{\Omega_0} q\delta w_0 dxdy - \int_{\Gamma} \Big(\hat{N}_{nn}\delta u_n + \hat{M}_{nn}\delta\phi_n - c_1 \hat{P}_{nn}\delta\varphi_n \\
& + \hat{N}_{ns}\delta u_s + \hat{M}_{ns}\delta\phi_s - c_1 \hat{P}_{ns}\delta\varphi_{ns} + \hat{Q}_n \delta w_0 \Big) d\Gamma
\end{aligned}
$$

(8.2-10)

$$
\begin{aligned}
\delta K &= \int_{\Omega_0}\int_{-\frac{h}{2}}^{\frac{h}{2}} \rho_0 \Big[\left(\dot{u}_0 + z\dot{\phi}_x - c_1 z^3 \dot{\varphi}_x\right)\left(\delta\dot{u}_0 + z\delta\dot{\phi}_x - c_1 z^3 \delta\dot{\varphi}_x\right) \\
&\qquad + \left(\dot{v}_0 + z\dot{\phi}_y - c_1 z^3 \dot{\varphi}_y\right)\left(\delta\dot{v}_0 + z\delta\dot{\phi}_y - c_1 z^3 \delta\dot{\varphi}_y\right) + \dot{w}_0 \delta\dot{w}_0 \Big] dv \\
&= \int_{\Omega_0} \Big[\left(I_0\dot{u}_0 + I_1\dot{\phi}_x - c_1 I_3 \dot{\varphi}_x\right)\delta\dot{u}_0 + \left(I_1\dot{u}_0 + I_2\dot{\phi}_x - c_1 I_4\dot{\varphi}_x\right)\delta\dot{\phi}_x \\
&\quad - c_1\left(I_3\dot{u}_0 + I_4\dot{\phi}_x - c_1 I_6\dot{\varphi}_x\right)\delta\dot{\varphi}_x + \left(I_0\dot{v}_0 + I_1\dot{\phi}_y - c_1 I_3\dot{\varphi}_y\right)\delta\dot{v}_0 \\
&\quad + \left(I_1\dot{v}_0 + I_2\dot{\phi}_y - c_1 I_4\dot{\varphi}_y\right)\delta\dot{\phi}_y - c_1\left(I_3\dot{u}_0 + I_4\dot{\phi}_y - c_1 I_6\dot{\varphi}_y\right)\delta\dot{\varphi}_y \Big] dxdy
\end{aligned}
\qquad (8.2\text{-}11)
$$

wher Ω_0 denotes the midplane of the laminate, and

$$
\begin{Bmatrix} N_{\alpha\beta} \\ M_{\alpha\beta} \\ P_{\alpha\beta} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha\beta} \begin{Bmatrix} 1 \\ z \\ z^3 \end{Bmatrix} dz, \quad \begin{Bmatrix} Q_\alpha \\ R_\alpha \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha z} \begin{Bmatrix} 1 \\ z^2 \end{Bmatrix} dz \qquad (8.2\text{-}12)
$$

$$
I_i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho_0 (z)^i \, dz \quad (i = 0, 1, 2, \cdots, 6)
$$

By using above equations the equations of motion can be obtain:

$$\frac{\partial N_{xx}}{\partial x}+\frac{\partial N_{xy}}{\partial y}=I_0\ddot{u}_0+J_1\ddot{\phi}_x-c_1I_3\frac{\partial\ddot{w}_0}{\partial x} \quad a$$

$$\frac{\partial N_{xy}}{\partial x}+\frac{\partial N_{yy}}{\partial y}=I_0\ddot{v}_0+J_1\ddot{\phi}_y-c_1I_3\frac{\partial\ddot{w}_0}{\partial y} \quad b$$

$$\frac{\partial\bar{Q}_x}{\partial x}+\frac{\partial\bar{Q}_y}{\partial y}+\frac{\partial}{\partial x}(N_{xx}\frac{\partial w_0}{\partial x}+N_{xy}\frac{\partial w_0}{\partial y})+\frac{\partial}{\partial y}(N_{xy}\frac{\partial w_0}{\partial x}+N_{yy}\frac{\partial w_0}{\partial y})$$

$$+c_1(\frac{\partial^2P_{xx}}{\partial x^2}+2\frac{\partial^2P_{xy}}{\partial x\partial y}+\frac{\partial^2P_{yy}}{\partial y^2})+q=I_0\ddot{w}_0-c_1^2I_6(\frac{\partial^2\ddot{w}_0}{\partial x^2}+\frac{\partial^2\ddot{w}_0}{\partial y^2}) \quad c$$

$$+c_1\left[I_3(\frac{\partial\ddot{u}_0}{\partial x}+\frac{\partial\ddot{v}_0}{\partial y})+J_4(\frac{\partial\ddot{\phi}_x}{\partial x}+\frac{\partial\ddot{\phi}_y}{\partial y})\right]$$

$$\frac{\partial\bar{M}_{xx}}{\partial x}+\frac{\partial\bar{M}_{xy}}{\partial y}-\bar{Q}_x=J_1\ddot{u}_0+K_2\ddot{\phi}_x-c_1J_4\frac{\partial\ddot{w}_0}{\partial x} \quad d$$

$$\frac{\partial\bar{M}_{xy}}{\partial x}+\frac{\partial\bar{M}_{yy}}{\partial y}-\bar{Q}_y=J_1\ddot{v}_0+K_2\ddot{\phi}_y-c_1J_4\frac{\partial\ddot{w}_0}{\partial y} \quad e$$

(8.2-13)

where

$$\bar{M}_{\alpha\beta}=M_{\alpha\beta}-c_1P_{\alpha\beta}\ (\alpha,\beta=1,2,6);\quad \bar{Q}_\alpha=Q_\alpha-c_2R_\alpha\ (\alpha=4,5)$$

$$I_i=\sum_{k=1}^{N}\int_{z_k}^{z_{k+1}}\rho^{(k)}(z)^i\,dz\ \ (i=0,1,2,\cdots,6)$$

$$J_i=I_i-c_1I_{i+2},\ K_2=I_2-2c_1I_4+c_1^2I_6,\ c_1=\frac{4}{3h^2},\ c_2=3c_1$$

(8.2-14)

Primary Variables: $u_n, u_s, w_0, \dfrac{\partial w_0}{\partial n}, \phi_n, \phi_s$ (8.2-15)

secondary variables: $N_n, N_{ns}, \bar{V}_n, P_{nn}, \bar{M}_{nn}, \bar{M}_{ns}$

where

$$
\begin{aligned}
\bar{V}_n \equiv\ & c_1\left[\left(\frac{\partial P_{xx}}{\partial x}+\frac{\partial P_{xy}}{\partial y}\right) n_x+\left(\frac{\partial P_{xy}}{\partial x}+\frac{\partial P_{yy}}{\partial y}\right) n_y\right] \\
& -c_1\left[\left(I_3 \ddot{u}_0+J_4 \ddot{\phi}_x-c_1 I_6 \frac{\partial \ddot{w}_0}{\partial x}\right) n_x+\left(I_3 \ddot{v}_0+J_4 \ddot{\phi}_y-c_1 I_6 \frac{\partial \ddot{w}_0}{\partial y}\right) n_y\right] \\
& +\left(\bar{Q}_x n_x+\bar{Q}_y n_y\right)+\mathcal{P}(w_0)+c_1 \frac{\partial P_{ns}}{\partial s} \\
\mathcal{P}(w_0) =\ & \left(N_{xx} \frac{\partial w_0}{\partial x}+N_{xy} \frac{\partial w_0}{\partial y}\right) n_x+\left(N_{xy} \frac{\partial w_0}{\partial x}+N_{yy} \frac{\partial w_0}{\partial y}\right) n_y
\end{aligned}
$$

(8.2-16)

we can write:

$$
\boxed{
\begin{aligned}
&\left\{\begin{array}{c}\{N\} \\ \{M\} \\ \{P\}\end{array}\right\}=\left[\begin{array}{ccc}[A] & [B] & [E] \\ {[B]} & [D] & [F] \\ {[E]} & [F] & [H]\end{array}\right]\left\{\begin{array}{c}\{\varepsilon^{(0)}\} \\ \{\varepsilon^{(1)}\} \\ \{\varepsilon^{(3)}\}\end{array}\right\} \\
&\left\{\begin{array}{c}\{Q\} \\ \{R\}\end{array}\right\}=\left[\begin{array}{cc}[A] & [D] \\ {[D]} & [F]\end{array}\right]\left\{\begin{array}{c}\{\gamma^{(0)}\} \\ \{\gamma^{(2)}\}\end{array}\right\}
\end{aligned}
}
$$

Constitutive equations of TSDT laminats

(8.2-17)

$$(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij}) = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \bar{Q}_{ij}^{(k)} \left(1, z, z^2, z^3, z^4, z^6\right) dz$$ $i,j=1,2,6$ (8.2-18)

matrices are of the order 3×3

$$(A_{ij}, D_{ij}, F_{ij}) = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \bar{Q}_{ij}^{(k)} \left(1, z^2, z^4\right) dz$$ $i,j=4,5$ (8.2-19)

matrices are of the order 2×2

we have known A, B and D matrices, but the additional stiffnes coefficients are difined by:

$$E_{ij} = \frac{1}{4} \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)} \left[(z_{k+1})^4 - (z_k)^4\right]$$

$$F_{ij} = \frac{1}{5} \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)} \left[(z_{k+1})^5 - (z_k)^5\right]$$ (8.2-20)

$$H_{ij} = \frac{1}{7} \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)} \left[(z_{k+1})^7 - (z_k)^7\right]$$

In equations (8.2-13) by considering $C_1=0$ we will obtain the FSDT geverning equations.

8.3 Higher-order Laminate Stiffness characteristics

A simplified third-order theory may be deduced from the general third-order theory presented here by omitting the higher-order stress resultants (P_{xx}, P_{yy}, P_{xy}) but keeping the higher-order stress resultants (R_x, R_y). The resulting theory is not consistent in energy sense.

$$Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}}, \quad Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}, \quad Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}}$$

$$Q_{66} = G_{12},\ Q_{44} = G_{23},\ Q_{55} = G_{13}$$

I. Single-Layer Plates

Single Isotropic Layer

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & \nu A_{11} & 0 \\ \nu A_{11} & A_{11} & 0 \\ 0 & 0 & \frac{1-\nu}{2} A_{11} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix}$$

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & \nu D_{11} & 0 \\ \nu D_{11} & D_{11} & 0 \\ 0 & 0 & \frac{1-\nu}{2} D_{11} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} + \begin{bmatrix} F_{11} & \nu F_{11} & 0 \\ \nu F_{11} & F_{11} & 0 \\ 0 & 0 & \frac{1-\nu}{2} F_{11} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(3)} \\ \varepsilon_{yy}^{(3)} \\ \gamma_{xy}^{(3)} \end{Bmatrix}$$

(11.3.4b) (8.2-22)

$$\begin{Bmatrix} P_{xx} \\ P_{yy} \\ P_{xy} \end{Bmatrix} = \begin{bmatrix} F_{11} & \nu F_{11} & 0 \\ \nu F_{11} & F_{11} & 0 \\ 0 & 0 & \frac{1-\nu}{2} F_{11} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} + \begin{bmatrix} H_{11} & \nu H_{11} & 0 \\ \nu H_{11} & H_{11} & 0 \\ 0 & 0 & \frac{1-\nu}{2} H_{11} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(3)} \\ \varepsilon_{yy}^{(3)} \\ \gamma_{xy}^{(3)} \end{Bmatrix}$$

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = \frac{1-\nu}{2} \begin{bmatrix} A_{11} & 0 \\ 0 & A_{11} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \end{Bmatrix} + \frac{1-\nu}{2} \begin{bmatrix} D_{11} & 0 \\ 0 & D_{11} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^{(2)} \\ \gamma_{xz}^{(2)} \end{Bmatrix}$$

$$\begin{Bmatrix} R_y \\ R_x \end{Bmatrix} = \frac{1-\nu}{2} \begin{bmatrix} D_{11} & 0 \\ 0 & D_{11} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \end{Bmatrix} + \frac{1-\nu}{2} \begin{bmatrix} F_{11} & 0 \\ 0 & F_{11} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^{(2)} \\ \gamma_{xz}^{(2)} \end{Bmatrix}$$

Single Specially Orthotropic Layer

similar equations hold for N's and M's) (8.2-22) and

$$\begin{Bmatrix} P_{xx} \\ P_{yy} \\ P_{xy} \end{Bmatrix} = \frac{h^5}{80} \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} + \frac{h^7}{448} \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(3)} \\ \varepsilon_{yy}^{(3)} \\ \gamma_{xy}^{(3)} \end{Bmatrix} \quad (11.3$$

(8.2-23)

$$\begin{Bmatrix} R_y \\ R_x \end{Bmatrix} = \frac{h^3}{12} \begin{bmatrix} Q_{44} & 0 \\ 0 & Q_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \end{Bmatrix} + \frac{h^5}{80} \begin{bmatrix} Q_{44} & 0 \\ 0 & Q_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^{(2)} \\ \gamma_{xz}^{(2)} \end{Bmatrix} \quad (11.3$$

Single Generally Orthotropic Layer

$$\begin{Bmatrix} P_{xx} \\ P_{yy} \\ P_{xy} \end{Bmatrix} = \begin{bmatrix} F_{11} & F_{12} & F_{16} \\ F_{12} & F_{22} & F_{26} \\ F_{16} & F_{26} & F_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} + \begin{bmatrix} H_{11} & H_{12} & H_{16} \\ H_{12} & H_{22} & H_{26} \\ H_{16} & H_{26} & H_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(3)} \\ \varepsilon_{yy}^{(3)} \\ \gamma_{xy}^{(3)} \end{Bmatrix}$$

(8.2-24)

$$\begin{Bmatrix} R_y \\ R_x \end{Bmatrix} = \begin{bmatrix} D_{44} & D_{45} \\ D_{45} & D_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \end{Bmatrix} + \begin{bmatrix} F_{44} & F_{45} \\ F_{45} & F_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^{(2)} \\ \gamma_{xz}^{(2)} \end{Bmatrix}$$

higher-order thermal stress resultants for this case are given by

$$\begin{Bmatrix} P_{xx}^T \\ P_{yy}^T \\ P_{xy}^T \end{Bmatrix} = \sum_{k=1}^{L} \int_{z_k}^{z_{k+1}} \begin{bmatrix} \bar{Q}_{11}^k & \bar{Q}_{12}^k & \bar{Q}_{16}^k \\ \bar{Q}_{12}^k & \bar{Q}_{22}^k & \bar{Q}_{26}^k \\ \bar{Q}_{16}^k & \bar{Q}_{26}^k & \bar{Q}_{66}^k \end{bmatrix}^{(k)} \begin{Bmatrix} \alpha_{xx} \\ \alpha_{yy} \\ 2\alpha_{xy} \end{Bmatrix}^{(k)} \Delta T z^3 dz$$

(8.2-25)

II Symmetric Laminates

similar equations hold for N's and M's) (8.2-22)

Symmetric Laminates with Multiple Isotropic Layers

$$\begin{Bmatrix} P_{xx} \\ P_{yy} \\ P_{xy} \end{Bmatrix} = \begin{bmatrix} F_{11} & F_{12} & 0 \\ F_{12} & F_{11} & 0 \\ 0 & 0 & F_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} + \begin{bmatrix} H_{11} & H_{12} & 0 \\ H_{12} & H_{11} & 0 \\ 0 & 0 & H_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(3)} \\ \varepsilon_{yy}^{(3)} \\ \gamma_{xy}^{(3)} \end{Bmatrix} \qquad (8.2\text{-}26)$$

$$\begin{Bmatrix} R_y \\ R_x \end{Bmatrix} = \begin{bmatrix} D_{44} & 0 \\ 0 & D_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \end{Bmatrix} + \begin{bmatrix} F_{44} & 0 \\ 0 & F_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^{(2)} \\ \gamma_{xz}^{(2)} \end{Bmatrix}$$

III Antisymmetric Laminates

$F_{16} = F_{26} = H_{16} = H_{26} = 0$ & $B_{ij}, F_{ij} \neq 0$

$$\begin{Bmatrix} P_{xx} \\ P_{yy} \\ P_{xy} \end{Bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{16} \\ E_{12} & E_{22} & E_{26} \\ E_{16} & E_{26} & E_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \varepsilon_{xy}^{(0)} \end{Bmatrix} + \begin{bmatrix} F_{11} & F_{12} & 0 \\ F_{12} & F_{22} & 0 \\ 0 & 0 & F_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \varepsilon_{xy}^{(1)} \end{Bmatrix} + \begin{bmatrix} H_{11} & H_{12} & 0 \\ H_{12} & H_{22} & 0 \\ 0 & 0 & H_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(3)} \\ \varepsilon_{yy}^{(3)} \\ \varepsilon_{xy}^{(3)} \end{Bmatrix} \qquad (8.2\text{-}27)$$

$$\begin{Bmatrix} R_y \\ R_x \end{Bmatrix} = \begin{bmatrix} D_{44} & 0 \\ 0 & D_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_{yz}^{(0)} \\ \varepsilon_{xz}^{(0)} \end{Bmatrix} + \begin{bmatrix} F_{44} & 0 \\ 0 & F_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_{yz}^{(2)} \\ \varepsilon_{xz}^{(2)} \end{Bmatrix}$$

8.4 The Navier Solutions

In fact, it is possible to develop the Navier Solutions of simply supported antisymmetric cross-ply and angle-ply laminates using the third-order theory. For antisimmetric cross-ply laminates the following stiffnesses are zero:

$$\begin{aligned}
&A_{16} = A_{26} = A_{45} = B_{16} = B_{26} = D_{16} = D_{26} = I_1 = 0 \\
&E_{16} = E_{26} = F_{16} = F_{26} = H_{16} = H_{26} = D_{45} = F_{45} = I_3 = I_5 = I_7 = 0
\end{aligned} \tag{8.4-1}$$

For antisimmetric angle-ply laminates the following stiffnesses are zero

$$\begin{aligned}
&A_{16} = A_{26} = A_{45} = B_{11} = B_{12} = B_{22} = B_{66} = D_{16} = D_{26} = I_1 = 0 \\
&E_{11} = E_{12} = E_{22} = E_{66} = F_{16} = F_{26} = H_{16} = H_{26} = D_{45} = F_{45} = I_3 = I_5 = I_7 = 0
\end{aligned} \tag{8.4-2}$$

For this approach the boundary conditions can be SS_1 or SS-2 :