Composites Lesson 24 بسم الله الرحمن الرحيم

7.4-2 Bending

$$A\frac{d^2u_0}{dx^2} = B\frac{d^3w_0}{dx^3} + A_{66}\frac{dN_{xx}^T}{dx} - A_{16}\frac{dN_{xy}^T}{dx}$$
$$A\frac{d^2v_0}{dx^2} = C\frac{d^3w_0}{dx^3} + A_{11}\frac{dN_{xy}^T}{dx} - A_{16}\frac{dN_{xx}^T}{dx}$$
$$D\frac{d^4w_0}{dx^4} = \bar{B}\frac{d^2N_{xx}^T}{dx^2} + \bar{C}\frac{d^2N_{xy}^T}{dx^2} - \frac{d^2M_{xx}^T}{dx^2} + q$$

(7.4-5 a,b,c)

Equation (7.4-5c) governing w_o is uncoupled from those governing (u_o, v_o).

When both edges are simply supported or clamped, exact solutions can be developed without any restrictions on the laminated scheme.

By integrating the third Eq. and substituting it into the other equations we have:

$$D\frac{d^3w_0}{dx^3} = \bar{B}\frac{dN_{xx}^T}{dx} + \bar{C}\frac{dN_{xy}^T}{dx} - \frac{dM_{xx}^T}{dx} + \int_0^x q(\xi)\ d\xi + c_1$$
$$A\frac{d^2u_0}{dx^2} = \hat{B}\int_0^x q(\xi)\ d\xi + G_1\frac{dN_{xx}^T}{dx} + F_1\frac{dN_{xy}^T}{dx} - \hat{B}\frac{dM_{xx}^T}{dx} + a_1 \qquad (7.4\text{-}6)$$
$$A\frac{d^2v_0}{dx^2} = \hat{C}\int_0^x q(\xi)\ d\xi + G_2\frac{dN_{xx}^T}{dx} + F_2\frac{dN_{xy}^T}{dx} - \hat{C}\frac{dM_{xx}^T}{dx} + b_1$$

where

$$G_1 = \frac{\bar{B}B}{D} + A_{66},\quad F_1 = \frac{\bar{B}B}{D} - A_{16},\quad \hat{B} = \frac{B}{D}$$
$$G_2 = \frac{\bar{B}C}{D} - A_{16},\quad F_2 = \frac{\bar{B}C}{D} + A_{11},\quad \hat{C} = \frac{C}{D} \qquad (7.4\text{-}7)$$
$$a_1 = \hat{B}c_1,\quad b_1 = \hat{C}c_1$$

Further integrations lead to

$$Au_0(x) = \hat{B}\int_0^x \left[\int_0^\xi \left(\int_0^\eta q(\zeta)\, d\zeta\right)\, d\eta\right] d\xi + G_1\int_0^x N_{xx}^T(\xi)d\xi + F_1\int_0^x N_{xy}^T(\xi)d\xi$$
$$-\hat{B}\int_0^x M_{xx}^T(\xi)d\xi + a_1\frac{x^2}{2} + a_2 x + a_3$$
$$Av_0(x) = \hat{C}\int_0^x \left[\int_0^\xi \left(\int_0^\eta q(\zeta)\, d\zeta\right)\, d\eta\right] d\xi + G_2\int_0^x N_{xx}^T(\xi)d\xi + F_2\int_0^x N_{xy}^T(\xi)d\xi$$
$$-\hat{C}\int_0^x M_{xx}^T(\xi)d\xi + b_1\frac{x^2}{2} + b_2 x + b_3$$
$$D\frac{dw_0}{dx} = \int_0^x \left[\int_0^\xi \left(\int_0^\eta q(\zeta)\, d\zeta\right)\, d\eta\right] d\xi + \bar{B}\int_0^x N_{xx}^T(\xi)d\xi + \bar{C}\int_0^x N_{xy}^T(\xi)d\xi$$
$$-\int_0^x M_{xx}^T(\xi)d\xi + c_1\frac{x^2}{2} + c_2 x + c_3$$
$$Dw_0(x) = \int_0^x \left\{\int_0^\xi \left[\int_0^\eta \left(\int_0^\zeta q(\mu)d\mu\right)\, d\zeta\right]\, d\eta\right\} d\xi + \bar{B}\int_0^x \left(\int_0^\xi N_{xx}^T(\eta)d\eta\right) d\xi$$
$$+\bar{C}\int_0^x \left(\int_0^\xi N_{xy}^T(\eta)d\eta\right) d\xi - \int_0^x \left(\int_0^\xi M_{xx}^T(\eta)d\eta\right) d\xi$$
$$+c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3 x + c_4$$

(7.4-8)

Example: Simply supported plate strip

For a plate strip with simply supported edges at $x = 0$ and $x = a$, the boundary conditions are (see Table (*))

$$N_{xx} = 0, \; w_0 = 0, \; M_{xx} = 0$$

where

$$N_{xx} = A_{11}\frac{du_0}{dx} + A_{16}\frac{dv_0}{dx} - B_{11}\frac{d^2w_0}{dx^2} - N_{xx}^T$$

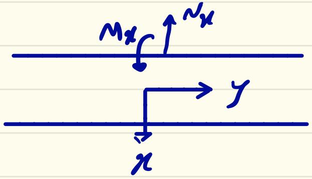
$$N_{yy} = A_{12}\frac{du_0}{dx} + A_{26}\frac{dv_0}{dx} - B_{12}\frac{d^2w_0}{dx^2} - N_{yy}^T$$

$$N_{xy} = A_{16}\frac{du_0}{dx} + A_{66}\frac{dv_0}{dx} - B_{16}\frac{d^2w_0}{dx^2} - N_{xy}^T$$

$$M_{xx} = B_{11}\frac{du_0}{dx} + B_{16}\frac{dv_0}{dx} - D_{11}\frac{d^2w_0}{dx^2} - M_{xx}^T$$

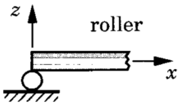
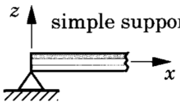
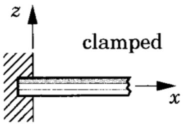
$$M_{yy} = B_{12}\frac{du_0}{dx} + B_{26}\frac{dv_0}{dx} - D_{12}\frac{d^2w_0}{dx^2} - M_{yy}^T$$

$$M_{xy} = B_{16}\frac{du_0}{dx} + B_{66}\frac{dv_0}{dx} - D_{16}\frac{d^2w_0}{dx^2} - M_{xy}^T$$



(*)

Table 4.4.1: Boundary conditions in the classical (CLPT) and first-order shear deformation (FSDT) theories of beams and plate strips. The boundary conditions on u_0 and v_0 are only for laminated strips in cylindrical bending.

Edge Condition	CLPT		FSDT	
 free	$N_{xx}=0$	$N_{xy}=0$	$N_{xx}=0$	$N_{xy}=0$
	$M_{xx}=0$	$\frac{dM_{xx}}{dx}=0$	$M_{xx}=0$	$Q_x=0$
 roller	$w_0=0$	$\frac{dv_0}{dx}=0$	$w_0=0$	$\frac{dv_0}{dx}=0$
	$N_{xx}=0$	$M_{xx}=0$	$N_{xx}=0$	$M_{xx}=0$
 simple support	$u_0=0$	$w_0=0$	$u_0=0$	$w_0=0$
	$\frac{dv_0}{dx}=0$	$M_{xx}=0$	$\frac{dv_0}{dx}=0$	$M_{xx}=0$
 clamped	$u_0=0$	$v_0=0$	$u_0=0$	$v_0=0$
	$w_0=0$	$\frac{dw_0}{dx}=0$	$w_0=0$	$\phi_x=0$

(7.4-8) $\rightarrow$

$$u_0(x) = \frac{B}{AD}\frac{q_0a^3}{12}\left[2\left(\frac{x}{a}\right)^3 - 3\left(\frac{x}{a}\right)^2\right] + \hat{N}_{xx}^T\, x + a_3$$

$$v_0(x) = \frac{C}{AD}\frac{q_0a^3}{12}\left[2\left(\frac{x}{a}\right)^3 - 3\left(\frac{x}{a}\right)^2\right] + b_3$$

$$w_0(x) = \frac{q_0a^4}{24D}\left[\left(\frac{x}{a}\right)^4 - 2\left(\frac{x}{a}\right)^3 + \left(\frac{x}{a}\right)\right] + \frac{\hat{M}_{xx}^T a^2}{2}\left[\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)\right]$$

The maximum transverse deflection occurs at $x = a/2$, and it is given by

$$w_{max} = \frac{5q_0a^4}{384D} - \frac{\hat{M}_{xx}^T a^2}{8}$$

In order to see the effect of the bending-stretching coupling on the transverse deflection, the reciprocal of the bending stiffness D [see Eq. (7.4-4)] is expressed as

$$\frac{1}{D} = \frac{1}{D_{11}}\left(\frac{D_{11}}{D}\right) = \frac{1}{D_{11}}\left(\frac{D + B_{11}\bar{B} + B_{16}\bar{C}}{D}\right)$$

$\Rightarrow$

$$w_{max} = \frac{5q_0a^4}{384D_{11}}\left(1 + \frac{B_{11}\bar{B} + B_{16}\bar{C}}{D}\right) - \frac{\hat{M}_{xx}^T a^2}{8}$$

1- For symmetric laminates the coupling terms are zero $(B=0)$

$$w_{max}=\frac{5q_0a^4}{384D_{11}}-\frac{\hat{M}^T_{xx}a^2}{8}$$

2- For example, for antisymmetric cross-ply laminates, we have $A_{16}=A_{26}=B_{16}=B_{26}=D_{16}=D_{26}=0$, $\bar{B}=B_{11}/A_{11}$, $\bar{C}=0$, and $D=D_{11}-B_{11}^2/A_{11}$. Thus the maximum deflection becomes

$$w_{max}=-\frac{5q_0a^4}{384D_{11}}\left(1+\frac{B_{11}^2}{A_{11}D_{11}-B_{11}^2}\right)-\frac{\hat{M}^T_{xx}a^2}{8}$$

3- In the case of antisymmetric angle-ply laminates, we have $A_{16}=A_{26}=B_{11}=B_{22}=B_{12}=B_{66}=D_{16}=D_{26}=0$, $\bar{B}=0$, $\bar{C}=B_{16}/A_{66}$, and $D=D_{11}-B_{16}^2/A_{66}$. The maximum deflection becomes

$$w_{max}=\frac{5q_0a^4}{384D_{11}}\left(1+\frac{B_{16}^2}{A_{66}D_{11}-B_{16}^2}\right)-\frac{\hat{M}^T_{xx}a^2}{8}$$

$(B=0)$

Note that when the bending-stretching coupling terms are zero (e.g., for symmetric laminates), the cylindrical bending and laminated beam solutions have the same form. The difference is only in the bending stiffness term. The bending stiffness D_{11} used in cylindrical bending is given by

$$D_{11}=\frac{E^b_{xx}h^3}{12(1-\nu^b_{xy}\nu^b_{yx})}=\frac{E^b_{xx}h^3}{12[1-(\nu^b_{xy})^2(E^b_{xx}/E^b_{yy})]}$$

whereas the bending stiffness used in the beam theory is $E^b_{xx}I_{yy}=E^b_{xx}bh^3/12$. Thus, the difference is in the expression containing Poisson's ratios, which is due to the plane strain assumption used

7.4-3 Buckling

The equilibrium of the plate strip under the applied in-plane compressive load $\hat{N}_x = -N_x^0$ can be obtained from Eq. (7.4-3) by omitting the inertia terms and termal resultants.

$$A\frac{d^2U}{dx^2} = B\frac{d^3W}{dx^3}$$
$$A\frac{d^2V}{dx^2} = C\frac{d^3W}{dx^3}$$
$$D\frac{d^4W}{dx^4} = -N_{xx}^0\frac{d^2W}{dx^2}$$

Buckling of plate strip by CLPT (7.4-9)

From the third equation we have:

$$D\frac{d^2w}{dx^2} + N_x^0 w = K_1 x + K_2 \qquad (7.4\text{-}10)$$

The general solution of (7.4-10) is:

$$W(x) = C_1 \sin \lambda x + C_2 \cos \lambda x + C_3 x + C_4 \qquad (7.4\text{-}11)$$

where

$$C_3 = K_1/\lambda^2 , \quad C_4 = K_2/\lambda^2 , \quad \lambda^2 = \frac{N_x^0}{D} \quad \text{or} \quad N_x^0 = D\lambda^2 \qquad (7.4\text{-}12)$$

Example:

When the plate strip is simply supported at $x = 0, a$, from Eq. (*) we have

$$W = 0, \quad \frac{dU}{dx} = 0, \quad \frac{dV}{dx} = 0, \quad \frac{d^2W}{dx^2} = 0$$

Use of the boundary conditions on W gives $c_2 = c_3 = c_4 = 0$ and the result

$$\sin \lambda a \equiv \sin(\sqrt{\frac{N_{xx}^0}{D}}) = 0, \quad \text{or} \quad N_{xx}^0 = D\left(\frac{n\pi}{a}\right)^2$$

The critical buckling load N_{cr} is given by $(n = 1)$

$$N_{cr} = D_{11}\frac{\pi^2}{a^2}\left(1 - \frac{B_{11}\bar{B} + B_{16}\bar{C}}{D_{11}A}\right)$$

Thus the effect of the bending-extensional coupling is to decrease the critical buckling load.

Recall from Section 4.2.3 that when both edges are clamped, λ is determined by solving the equation

$$\lambda a \ \sin \lambda a + 2\cos \lambda a - 2 = 0$$

The smallest root of this equation is $\lambda = 2\pi$, and the critical buckling load becomes

$$N_{cr} = D_{11}\frac{4\pi^2}{a^2}\left(1 - \frac{B_{11}\bar{B} + B_{16}\bar{C}}{D_{11}A}\right)$$

7.4-4 Vibration

For vibration in the absence of in-plane inertias, thermal forces, and transverse load, Eq. (7.4-3) is reduced to

$$D\frac{\partial^4 w_0}{\partial x^4} = \bar{I}_2\frac{\partial^4 w_0}{\partial x^2 \partial t^2} - I_0\frac{\partial^2 w_0}{\partial t^2} + \hat{N}_{xx}\frac{\partial^2 w_0}{\partial x^2} \qquad (7.4\text{-}13)$$

where $\bar{I}_2 = I_2 - \bar{B} I_1$. For aperiodic motion, we assume

$$w_0(x,t) = W(x)\, e^{i\omega t}, \quad i = \sqrt{-1} \qquad (7.4\text{-}14)$$

$$\Rightarrow \quad D\frac{d^4 W}{dx^4} - \hat{N}_x \frac{d^2 W}{dx^2} = I_0 \omega^2 W - \bar{I}_2 \omega^2 \frac{d^2 W}{dx^2} \qquad (7.4\text{-}15)$$

General Solution:

$$W(x) = C_1 \sin(\lambda x) + C_2 \cos(\lambda x) + C_3 \sinh(\mu x) + C_4 \cosh(\mu x) \qquad (7.4\text{-}16)$$

where

$$\lambda = \sqrt{\frac{1}{2p}\left(q + \sqrt{q^2 + 4pr}\right)}, \quad \mu = \sqrt{\frac{1}{2p}\left(-q + \sqrt{q^2 + 4pr}\right)}$$

$$p = D, \quad q = \bar{I}_2\omega^2 - \hat{N}_{xx}, \quad r = I_0\omega^2$$

If the applied axial force is zero, the natural frequency of vibration, with rotary inertia included, is given by

$$\omega^2 = \lambda^4 \frac{D}{I_0}\left(1 - \frac{\bar{I}_2\lambda^2}{I_0 + \bar{I}_2\lambda^2}\right) \qquad (7.4\text{-}18)$$

When rotary inertia is neglected, we have

$$\omega = \lambda^2 \sqrt{\frac{D}{I_0}} \qquad (7.4\text{-}19)$$

Example:

For a simply supported plate strip, λ_n is given by $\lambda_n = \frac{n\pi}{a}$ and from Eq. (7.4-18) it follows that

$$\omega_n = \left(\frac{n\pi}{a}\right)^2 \sqrt{\frac{D}{I_0}} \sqrt{\frac{1}{1+(\frac{n\pi}{a})^2(\bar{I}_2/I_0)}}$$

Note that the rotary inertia has the effect of decreasing the natural frequency. When the rotary inertia is zero, we have

$$\omega_n = \left(\frac{n\pi}{a}\right)^2 \sqrt{\frac{D}{I_0}}$$

For a plate strip clamped at both ends, λ must be determined

$$-2+2\cos\lambda a\cosh\mu a + \left(\frac{\lambda}{\mu}-\frac{\mu}{\lambda}\right)\sin\lambda a\sinh\mu a = 0$$

For natural vibration without rotary inertia, above Eq takes the simpler form

$$\cos\lambda a\cosh\lambda a - 1 = 0$$

The roots of Eq. above are

$$\lambda_1 a = 4.730, \quad \lambda_2 a = 7.853, \quad \lambda_3 a = 10.996, \quad \cdots, \quad \lambda_n a \approx (n+\frac{1}{2})\pi$$

7.5 Cylindrical Bending Using FSDT

7.5_1 Governing Equations

For finding governing equations of FSDT plate strip consider $\frac{\partial}{\partial y} = 0$

$$A_{11}\frac{\partial^2 u_0}{\partial x^2} + A_{16}\frac{\partial^2 v_0}{\partial x^2} + B_{11}\frac{\partial^2 \phi_x}{\partial x^2} + B_{16}\frac{\partial^2 \phi_y}{\partial x^2} - \frac{\partial N_{xx}^T}{\partial x} = I_0\frac{\partial^2 u_0}{\partial t^2} + I_1\frac{\partial^2 \phi_x}{\partial t^2} \quad a$$

$$A_{16}\frac{\partial^2 u_0}{\partial x^2} + A_{66}\frac{\partial^2 v_0}{\partial x^2} + B_{16}\frac{\partial^2 \phi_x}{\partial x^2} + B_{66}\frac{\partial^2 \phi_y}{\partial x^2} - \frac{\partial N_{xy}^T}{\partial x} = I_0\frac{\partial^2 v_0}{\partial t^2} + I_1\frac{\partial^2 \phi_y}{\partial t^2} \quad b$$

$$B_{11}\frac{\partial^2 u_0}{\partial x^2} + B_{16}\frac{\partial^2 v_0}{\partial x^2} + D_{11}\frac{\partial^2 \phi_x}{\partial x^2} + D_{16}\frac{\partial^2 \phi_y}{\partial x^2} - KA_{55}\left(\frac{\partial w_0}{\partial x} + \phi_x\right) - KA_{45}\phi_y - \frac{\partial M_{xx}^T}{\partial x} = I_2\frac{\partial^2 \phi_x}{\partial t^2} + I_1\frac{\partial^2 u_0}{\partial t^2} \quad c$$

$$B_{16}\frac{\partial^2 u_0}{\partial x^2} + B_{66}\frac{\partial^2 v_0}{\partial x^2} + D_{16}\frac{\partial^2 \phi_x}{\partial x^2} + D_{66}\frac{\partial^2 \phi_y}{\partial x^2} - KA_{44}\phi_y - KA_{45}\left(\frac{\partial w_0}{\partial x} + \phi_x\right) - \frac{\partial M_{xy}^T}{\partial x} = I_2\frac{\partial^2 \phi_y}{\partial t^2} + I_1\frac{\partial^2 v_0}{\partial t^2} \quad d$$

$$KA_{55}\left(\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial \phi_x}{\partial x}\right) + KA_{45}\frac{\partial \phi_y}{\partial x} + \frac{\partial}{\partial x}\left(N_{xx}\frac{\partial w_0}{\partial x}\right) + q = I_0\frac{\partial^2 w_0}{\partial t^2} \quad e$$

Governing eq. of FSDT plate strip (7.5-1)

For cylindrical bending we further assume that $\phi_y = 0$ everywhere, and omit Eq. (7.5-1d) from further consideration. For the purpose of developing analytical solutions, we neglect the in-plane inertia terms and assume that there are no thermal effects.

$$A_{11}\frac{\partial^2 u_0}{\partial x^2} + A_{16}\frac{\partial^2 v_0}{\partial x^2} + B_{11}\frac{\partial^2 \phi_x}{\partial x^2} = I_1\frac{\partial^2 \phi_x}{\partial t^2}$$

$$A_{16}\frac{\partial^2 u_0}{\partial x^2} + A_{66}\frac{\partial^2 v_0}{\partial x^2} + B_{16}\frac{\partial^2 \phi_x}{\partial x^2} = 0$$

$$B_{11}\frac{\partial^2 u_0}{\partial x^2} + B_{16}\frac{\partial^2 v_0}{\partial x^2} + D_{11}\frac{\partial^2 \phi_x}{\partial x^2} - KA_{55}\left(\frac{\partial w_0}{\partial x} + \phi_x\right) = I_2\frac{\partial^2 \phi_x}{\partial t^2}$$

$$KA_{55}\left(\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial \phi_x}{\partial x}\right) + \frac{\partial}{\partial x}\left(N_{xx}\frac{\partial w_0}{\partial x}\right) + q = I_0\frac{\partial^2 w_0}{\partial t^2}$$

(7.5-2)

By omitting u_0 and v_0 from Eqs. (7.5-2) we have:

$$\boxed{\begin{aligned} KA_{55}\left(\frac{\partial^2 w_0}{\partial x^2}+\frac{\partial \phi_x}{\partial x}\right)+\hat{N}_{xx}\frac{\partial^2 w_0}{\partial x^2}+q=I_0\frac{\partial^2 w_0}{\partial t^2} \\ D\frac{\partial^2 \phi_x}{\partial x^2}-KA_{55}\left(\frac{\partial w_0}{\partial x}+\phi_x\right)=I_2\frac{\partial^2 \phi_x}{\partial t^2}\end{aligned}}$$

(7.5-3)

7.5-2 Bending

(7.5-3) $\Rightarrow$

$$KA_{55}\left(\frac{d^2 w_0}{dx^2}+\frac{d\phi_x}{dx}\right)+q=0$$

$$D\frac{d^2\phi_x}{dx^2}-KA_{55}\left(\frac{dw_0}{dx}+\phi_x\right)=0$$

Bending eqs. of FSDT plate strips (7.5-4)

General solution:

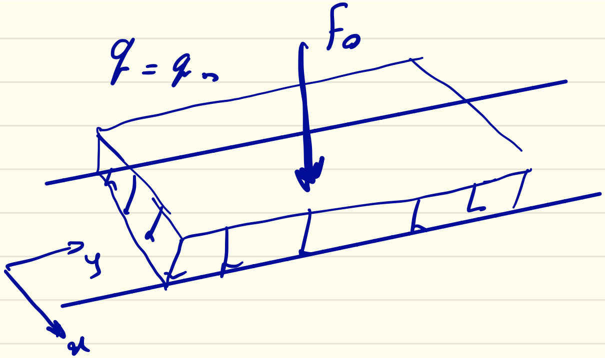
$$\phi_x(x)=\frac{1}{D}\left[-\int_0^x\int_0^\xi\int_0^\eta q(\zeta)\, d\zeta d\eta d\xi+c_1\frac{x^2}{2}+c_2x+c_3\right]$$

(7.5-5)

$$\begin{aligned} w_0(x)=&-\frac{1}{D}\left[-\int_0^x\int_0^\xi\int_0^\eta\int_0^\mu q(\zeta)\, d\zeta d\mu d\eta d\xi+c_1\frac{x^3}{6}+c_2\frac{x^2}{2}+c_3x+c_4\right] \\ &+\frac{1}{KA_{55}}\left[-\int_0^x\int_0^\xi q(\zeta)d\zeta d\xi+c_1x\right] \qquad (4\end{aligned}$$

(7.5-6)

Example: For a plate strip simply supported at both ends and subjected to uniformly distributed load $q = q_0$ as well as a downward point load F_0 at the center, we obtain



$$\phi_x(x) = -\frac{q_0 a^3}{24D}\left[4\left(\frac{x}{a}\right)^3 - 6\left(\frac{x}{a}\right)^2 + 1\right] + \frac{F_0 a^2}{16D}\left[1 - 4\left(\frac{x}{a}\right)^2\right]$$

$$w_0(x) = \frac{q_0 a^4}{24D}\left[\left(\frac{x}{a}\right)^4 - 2\left(\frac{x}{a}\right)^3 + \left(\frac{x}{a}\right)\right] + \frac{q_0 a^2}{2KA_{55}}\left[\left(\frac{x}{a}\right) - \left(\frac{x}{a}\right)^2\right]$$
$$+ \frac{F_0 a^3}{48D}\left[3\left(\frac{x}{a}\right) - 4\left(\frac{x}{a}\right)^3\right] + \frac{F_0 a}{2KA_{55}}\left(\frac{x}{a}\right)$$

The maximum deflection occurs at $x = a/2$ and it is given by

$$w_{max} = \left(\frac{5q_0 a^4}{384D} + \frac{q_0 a^2}{8KA_{55}} + \frac{F_0 a^3}{48D} + \frac{F_0 a}{4KA_{55}}\right)$$

Example: Clamed plate strip. $q = q_0$, $F = F_0$

$$\phi_x(x) = -\frac{q_0 a^3}{12D}\left[2\left(\frac{x}{a}\right)^3 - 3\left(\frac{x}{a}\right)^2 + \left(\frac{x}{a}\right)\right] - \frac{F_0 a^2}{8D}\left[2\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)\right]$$

$$w_0(x) = \frac{q_0 a^4}{24D}\left[\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)\right]^2 + \frac{q_0 a^2}{2KA_{55}}\left[\left(\frac{x}{a}\right) - \left(\frac{x}{a}\right)^2\right] + \frac{F_0 a^3}{48D}\left[3\left(\frac{x}{a}\right)^2 - 4\left(\frac{x}{a}\right)^3\right] + \frac{F_0 a}{2KA_{55}}\left(\frac{x}{a}\right)$$

The maximum deflection is given by

$$w_{max} = \left(\frac{q_0 a^4}{384D} + \frac{q_0 a^2}{8KA_{55}} + \frac{F_0 a^3}{192D} + \frac{F_0 a}{4KA_{55}}\right)$$

$\tau_{xz} = \frac{Q_x}{A}$

$K = \frac{U^c}{U^{FSDT}}$

The determination of the shear correction coefficient K for laminated structures is still an unresolved issue. Values of K for various special cases are available in the literature (see [4–8]). The most commonly used value of $K = 5/6$ is based on homogeneous, isotropic plates (see Section 3.4), although K depends, in general, on the lamination scheme, geometry, and material properties.

7.5-3 Buckling

For stability analysis, we set $q=0$, $\hat{N}_x = -N_x^0$ and $I_0 = I_2 = 0$ in Eq. (7.5-3)

$$K A_{55}\left(\frac{d^2W}{dx^2}+\frac{d\mathcal{X}}{dx}\right)+\hat{N}_{xx}\frac{d^2W}{dx^2}=0$$

$$D\frac{d^2\mathcal{X}}{dx^2}-KA_{55}\left(\frac{dW}{dx}+\mathcal{X}\right)=0$$

Buckling of FSDT plate strip (7.5-7)

$\underset{=}{a}>$

$$\frac{d\mathcal{X}}{dx}=-\left(1-\frac{N_{xx}^0}{KA_{55}}\right)\frac{d^2W}{dx^2} \quad (7.5\text{-}8)$$

$\Rightarrow$

$$\mathcal{X}(x)=-\left(1-\frac{N_{xx}^0}{KA_{55}}\right)\frac{dW}{dx}+K_1 \quad (7.5\text{-}9)$$

$\underset{\Rightarrow}{b}$

$$D\left(1-\frac{N_{xx}^0}{KA_{55}}\right)\frac{d^4W}{dx^4}+N_{xx}^0\frac{d^2W}{dx^2}=0 \quad (7.5\text{-}10)$$

General solution:

$$W(x)=c_1\sin\lambda x+c_2\cos\lambda x+c_3x+c_4 \quad (7.5\text{-}11)$$

where

$$\lambda^2 = \frac{N_{xx}^0}{\left(1 - \frac{N_{xx}^0}{KA_{55}}\right) D} \quad \text{or} \quad N_{xx}^0 = \frac{\lambda^2 D}{\left(1 + \frac{\lambda^2 D}{KA_{55}}\right)} \qquad (7.5\text{-}12)$$

Example:

For a simply supported plate strip, the critical buckling load is given by

$$N_{cr} = \left(\frac{\pi}{a}\right)^2 D \left[1 - \frac{D\left(\frac{\pi}{a}\right)^2}{KA_{55} + D\left(\frac{\pi}{a}\right)^2}\right]$$

Thus, the effect of the transverse shear deformation is to decrease the buckling load. Omission of the transverse shear deformation in the classical theory amounts to assuming infinite rigidity in the transverse direction (i.e., $A_{55} = G_{13} = \infty$); hence, in the classical laminate theory the structure is represented stiffer than it is.

Example

For a plate strip fixed at both ends, λ is governed by the equation

$$2(\cos \lambda a - 1)\left(1 + \frac{\lambda^2 D}{KA_{55}}\right) + \lambda a \sin \lambda a = 0$$

$$\Rightarrow \quad N_{cr} = \left(\frac{2\pi}{a}\right)^2 D \left[1 - \frac{D\left(\frac{2\pi}{a}\right)^2}{KA_{55} + D\left(\frac{2\pi}{a}\right)^2}\right]$$

The effect of shear deformation is significant for $a/h \leq 10$ in the case of simply supported boundary conditions, and $a/h \leq 20$ in the case of clamped boundary conditions.

7.5-4 vibration

For a periodic motion, we assume solution in the form

$$w_0(x,t) = W(x)\, e^{i\omega t} \quad , \quad \phi_x(x,t) = \mathcal{X}(x)\, e^{i\omega t} \quad , \; i = \sqrt{-1}$$

where ω is the natural frequency of vibration, and $W(x)$ and $\mathcal{X}(x)$ are the mode shapes. Substituting of the above solution forms into Eqs. (7.5-3) yields:

$$D\frac{d^2\mathcal{X}}{dx^2} - KA_{55}\left(\frac{dW}{dx} + \mathcal{X}\right) + I_2\omega^2\mathcal{X} = 0$$

$$KA_{55}\left(\frac{d^2W}{dx^2} + \frac{d\mathcal{X}}{dx}\right) + I_0\omega^2 W = 0 \qquad (7.5\text{-}13)$$

$$\Rightarrow \quad p\frac{d^4W}{dx^4} + q\frac{d^2W}{dx^2} - rW = 0 \qquad (7.5\text{-}14)$$

where

$$p = D, \quad q = \frac{I_0 D}{KA_{55}}\omega^2, \quad r = I_0\omega^2 \qquad (7.5\text{-}15)$$

General solution:

$$W(x) = c_1 \sin\lambda x + c_2 \cos\lambda x + c_3 \sinh\mu x + c_4 \cosh\mu x \qquad (7.5\text{-}16)$$

where

$$\lambda = \sqrt{\frac{1}{2p}\left(q + \sqrt{q^2 + 4pr}\right)}, \quad \mu = \sqrt{\frac{1}{2p}\left(-q + \sqrt{q^2 + 4pr}\right)} \qquad (7.5\text{-}17)$$

The frequencies ω can be determind from

$$(\omega^2)_1 = \frac{1}{2P}\left(Q - \sqrt{Q^2 - 4PR}\right), \quad (\omega^2)_2 = \frac{1}{2P}\left(Q + \sqrt{Q^2 - 4PR}\right) \qquad (7.5\text{-}18)$$

where

$$P = \frac{I_2}{KA_{55}}, \quad Q = \left[1 + \left(\frac{D}{KA_{55}} + \frac{I_2}{I_0}\right)\lambda^2\right], \quad R = \left(\frac{D}{I_0}\right)\lambda^4 \qquad (7.5\text{-}19)$$

when the rotary inertia is neglected, we have $P=0$ and the frequency is given by

$$\omega^2 = \frac{\bar{Q}}{R}, \quad \bar{Q} = \left[1 + \left(\frac{D}{KA_{55}}\right)\lambda^2\right], \quad R = \left(\frac{D}{I_0}\right)\lambda^4 \qquad (7.5\text{-}20)$$

Example:

For a simply supported plate strip, the boundary conditions give $C_2 = C_3 = C_4 = 0$ and,

$$\sin(\lambda a) = 0 \longrightarrow \lambda_n = \frac{n\pi}{a}$$

Substituting this λ into Eq. (7.5-18) gives two frequencies for each value of λ. When the rotary inertia is neglected we obtain from Eq. (7.5-20) the result

$$\omega_n = \left(\frac{n\pi}{a}\right)^2 \sqrt{\frac{D}{I_0}} \sqrt{\frac{KA_{55}}{KA_{55} + (\frac{n\pi}{a})^2 D}}$$

By neglecting the shear deformation ($A_{55} = G_{13} = \infty$) we obtain the result

$$\omega_n = \left(\frac{n\pi}{a}\right)^2 \sqrt{\frac{D}{I_0}}$$

which is the same as in CLPT. Thus, the effect of shear deformation is to reduce the frequency of natural vibration.