Composites Lesson 23 بسم الله الرحمن الرحيم

Let us denote the first expression in (II) by

$$\frac{dw_0^b}{dx} = \frac{F_0 b a^2}{16 E_x^b I_y}\left[1 - 4\left(\frac{x}{a}\right)^2\right] = -\phi_x \qquad (III)$$

From (7.3-15) we can say:

$$w_0 = w_0^b + w_0^s \longrightarrow \frac{dw_0}{dx} = \frac{dw_0^b}{dx} + \frac{dw_0^s}{dx} \qquad (IV)$$

$$\xrightarrow{IV} \frac{dw_0^s}{dx} = \frac{dw_0}{dx} - \frac{dw_0^b}{dx} = \frac{dw_0}{dx} + \phi_x$$

we knew

$$\gamma_{xz} = 2\varepsilon_{xz} = \frac{\partial w_0}{\partial x} + \phi_x$$

$$\Rightarrow \frac{dw_0^s}{dx} = \gamma_{xz}$$

$$II \Rightarrow \left.\frac{dw_0}{dx}\right|_{x=a/2} = \frac{F_0 b}{2 K G_{xz}^b\, b h} \qquad (V)$$

$$\left.\frac{dw_0^b}{dx}\right|_{x=a/2} = -\phi_x\left(\frac{a}{2}\right) = 0 \qquad (VI)$$

Note that, in contrast to the classical beam theory, the slope $\frac{dw_0}{dx}$ at the center of the beam in the Timoshenko beam theory is nonzero. However $\frac{dw_0^b}{dx} = -\phi_x$ is zero at $x = \frac{a}{2}$.

Integrating (II) with respect to x, we arrive at the expression:

$$\underbrace{w_0(x) = \frac{F_0 b a^3}{48 E_x^b I_y}\left[3\left(\frac{x}{a}\right) - 4\left(\frac{x}{a}\right)^3\right]}_{w_0^b \equiv w_0^{CLPT}} + \frac{F_0 b a}{2K G_{xz}^b bh}\left(\frac{x}{a}\right) \quad (\text{VII})$$

The maximum deflection occurs at $x = \frac{a}{2}$ and it is given by

$$\begin{aligned} w_{max} &= \frac{F_0 b a^3}{48 E_{xx}^b I_{yy}} + \frac{F_0 b a}{4K G_{xz}^b bh} \\ &= \frac{F_0 b a^3}{48 E_{xx}^b I_{yy}}\left[1 + \left(\frac{E_{xx}^b}{K G_{xz}^b}\right)\left(\frac{h}{a}\right)^2\right] \end{aligned} \quad (\text{VIII})$$

Equation (VIII) shows that the effect of shear deformation is to increase the deflection. The contribution due to shear deformation to the deflection depends on the modulus ratio $\frac{E_x^b}{G_{xz}^b}$ as well as the ratio of thickness to length $\frac{h}{a}$.

The effect of shear deformation is negligible for thin and long beams.

7.3-3 Buckling

For bouckling analysis, the inertia terms and applied transverse load q in Eq. (7.3-11) are set to zero to obtain the governing equations of buckling under compressive edge load $\hat{N}_x = -N_x^0$:

$$KG^b_{xz}bh\left(\frac{d^2W}{dx^2} + \frac{d\mathcal{X}}{dx}\right) + b\hat{N}_{xx}\frac{d^2W}{dx^2} = 0$$

$$E^b_{xx}I_{yy}\frac{d^2\mathcal{X}}{dx^2} - KG^b_{xz}bh\left(\frac{dW}{dx} + \mathcal{X}\right) = 0$$

Buckling of FSDT beam (7.3-19 a,b)

$\mathcal{X} \equiv \Phi_x$

$$a \Rightarrow \quad KG^b_{xz}bh\frac{d\mathcal{X}}{dx} = -\left(KG^b_{xz}bh - bN_x^0\right)\frac{d^2w}{dx^2} \quad (7.3\text{-}20)$$

$$\rightarrow KG^b_{xz}bh\,\mathcal{X}(x) = -\left(KG^b_{xz}bh - bN_x^0\right)\frac{dw}{dx} + k_1 \quad (7.3\text{-}21)$$

Next differentiate Eq. (7.3.19b) with respect to x

and substitute for $\frac{d\chi}{dx}$ from Eq. (7.3-20) to obtain the result:

$$E^b_{xx} I_{yy} \left(1 - \frac{bN^0_{xx}}{KG^b_{xz}bh}\right) \frac{d^4W}{dx^4} + bN^0_{xx}\frac{d^2W}{dx^2} = 0 \qquad (7.3\text{-}22)$$

The general solution of Eq. (7.3-22) is

$$W(x) = C_1 \sin \lambda x + C_2 \cos \lambda x + C_3 x + C_4 \qquad (7.3\text{-}23)$$

where

$$\lambda^2 = \frac{bN^0_{xx}}{\left(1 - \frac{bN^0_{xx}}{KG^b_{xz}bh}\right) E^b_{xx} I_{yy}} \quad \text{or} \quad bN^0_{xx} = \frac{\lambda^2 E^b_{xx} I_{yy}}{\left(1 + \frac{\lambda^2 E^b_{xx} I_{yy}}{KG^b_{xz}bh}\right)} \qquad (7.3\text{-}24)$$

Example: Simply supported beam.

b.c. :

$$W(0)=0,\quad W(a)=0,\quad \frac{d\mathcal{X}}{dx}(0)=0,\quad \frac{d\mathcal{X}}{dx}(a)=0$$

In view of Eq. (7.3-20) the above conditions are equivalent to

$$W(0)=0,\quad W(a)=0,\quad \frac{d^2W}{dx^2}(0)=0,\quad \frac{d^2W}{dx^2}(a)=0$$

The boundary conditions in Eq. above lead to the result $c_2=c_3=c_4=0$, and for $c_1\neq 0$ the requirement

$$\sin\lambda a=0 \quad \text{implies} \quad \lambda a=n\pi$$

Substituting for λ from Eq. (7.3-24) into Eq. above for N^0_{xx}, we obtain

$$\begin{aligned} bN^0_{xx} &= E^b_{xx}I_{yy}\left(\frac{n\pi}{a}\right)^2\left[\frac{KG^b_{xz}bh}{KG^b_{xz}bh+E^b_{xx}I_{yy}\left(\frac{n\pi}{a}\right)^2}\right] \\ &= E^b_{xx}I_{yy}\left(\frac{n\pi}{a}\right)^2\left[1-\frac{E^b_{xx}I_{yy}\left(\frac{n\pi}{a}\right)^2}{KG^b_{xz}bh+E^b_{xx}I_{yy}\left(\frac{n\pi}{a}\right)^2}\right] \end{aligned}$$

The critical buckling load is given by the minimum $(n=1)$

$$bN_{cr}=E^b_{xx}I_{yy}\left(\frac{\pi}{a}\right)^2\left[1-\frac{E^b_{xx}I_{yy}\left(\frac{\pi}{a}\right)^2}{KG^b_{xz}bh+E^b_{xx}I_{yy}\left(\frac{\pi}{a}\right)^2}\right]$$

7.3.4 Vibration

For natural vibration, we assume that the applied axial force and transverse load are zero and that the motion is periodic.

Equations (7.3-11) thake the form $w_0 = W(x)\, e^{i\omega t}$

$$KG^b_{xz} bh \left(\frac{d^2 W}{dx^2} + \frac{d\mathcal{X}}{dx} \right) + \omega^2 \hat{I}_0 W = 0$$

$$E^b_{xx} I_{yy} \frac{d^2 \mathcal{X}}{dx^2} - KG^b_{xz} bh \left(\frac{dW}{dx} + \mathcal{X} \right) + \omega^2 \hat{I}_2 \mathcal{X} = 0$$

Vibration of FSDT beam (7.3-25 a,b)

$\mathcal{X} \equiv \phi_x$

we use the same procedure as before to eliminate $\mathcal{X}$

(a) $\rightarrow$ $$KG_{xz}^b bh\frac{d\mathcal{X}}{dx} = -\hat{I}_0\omega^2 W - KG_{xz}^b bh\frac{d^2W}{dx^2}$$ (7.3-26)

$\Rightarrow$ $$E_{xx}^b I_{yy}\frac{d^4W}{dx^4} + \left(\frac{E_{xx}^b I_{yy}\hat{I}_0}{KG_{xz}^b bh} + \hat{I}_2\right)\omega^2\frac{d^2W}{dx^2} - \left(1 - \frac{\omega^2\hat{I}_2}{KG_{xz}^b bh}\right)\hat{I}_0\omega^2 W = 0$$ (7.3-27)

General form
$$p\frac{d^4W}{dx^4} + q\frac{d^2W}{dx^2} - rW = 0$$ (7.3-28)

where

$$p = E_{xx}^b I_{yy}, \quad q = \left(\frac{E_{xx}^b I_{yy}}{KG_{xz}^b bh} + \frac{\hat{I}_2}{\hat{I}_0}\right)\hat{I}_0\omega^2, \quad r = \left(1 - \frac{\omega^2\hat{I}_2}{KG_{xz}^b bh}\right)\hat{I}_0\omega^2$$ (7.3-29)

The general solution of Eq. (7.2_28)

$$w(x) = C_1 \sin\lambda x + C_2 \cos\lambda x + C_3 \sinh(\mu x) + C_4 \cosh(\mu x)$$ (7.3-30)

$$\lambda = \sqrt{\frac{1}{2p}(q+\sqrt{q^2+4pr}}, \quad \mu = \sqrt{\frac{1}{2p}(-q+\sqrt{q^2+4pr}}$$ (7.3-31)

Note that we have:

$$(2\lambda^2 p - q)^2 = q^2 + 4pr \quad \text{or} \quad p\lambda^4 - q\lambda^2 - r = 0$$ (7.3-32)

Alternatively, Eq. (7.3-28) can be written, with w given by Eq. (7.3-30) in terms of ω as:

$$P\omega^4 - Q\omega^2 + R = 0 \qquad (7.3\text{-}33)$$

where

$$P = \frac{\hat{I}_2}{KG_{xz}^b bh}, \quad Q = \left[1 + \left(\frac{E_{xx}^b I_{yy}}{KG_{xz}^b bh} + \frac{\hat{I}_2}{\hat{I}_0}\right)\lambda^2\right], \quad R = \left(\frac{E_{xx}^b I_{yy}}{\hat{I}_0}\right)\lambda^4 \qquad (7.3\text{-}34)$$

Hence, there are two (sets of) roots of this equation (when $\hat{I}_2 \neq 0$)

$$(\omega^2)_1 = \frac{1}{2P}\left(Q - \sqrt{Q^2 - 4PR}\right), \quad (\omega^2)_2 = \frac{1}{2P}\left(Q + \sqrt{Q^2 - 4PR}\right) \qquad (7.3\text{-}35)$$

It can be shown that $Q^2 - 4PR > 0$ (and $PQ > 0$), and therefore the frequency given by the first equa. is the smaller of the two values. When the rotary inertia is neglegible, we have $P=0$, and the frequency is given by:

$$\omega^2 = \frac{R}{\bar{Q}}, \quad \bar{Q} = \left[1 + \left(\frac{E_x^b I_y}{k G_{xz}^b bh}\right)\lambda^2\right], \quad R = \left(\frac{E_x^b I_y}{\hat{I}_o}\right)\lambda^4 \qquad (7.3\text{-}36)$$

Example: Simply supported beam

The boundary conditions yield $C_2 = C_3 = C_4 = 0$ and

$c_1 \sin \lambda a = 0$, which implies $\lambda_n = \frac{n\pi}{a}$

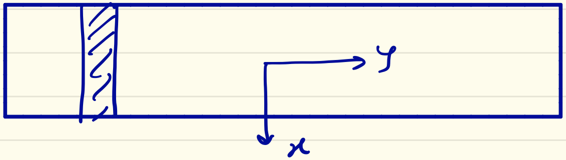
substitution of λ into Eq. (7.3-36) and the result into Eq. (7.3-35) gives two frequencies for each value of λ. The fundamental frequency will came from Eq. (7.3-35a).

when the rotary inertia is neglegible, we obtain from Eq. (7.3-36) the result:

$$\omega_n = \left(\frac{n\pi}{a}\right)^2 \sqrt{\frac{E^b_{xx} I_{yy}}{\hat{I}_0}} \sqrt{1 - \frac{(\frac{n\pi}{a})^2 E^b_{xx} I_{yy}}{K G^b_{xz} b h + (\frac{n\pi}{a})^2 E^b_{xx} I_{yy}}}$$

7.4 - Cylindrical Bending Using CLPT

7.4-1 Governing Equations



$q(x)$, $\frac{\partial}{\partial y} = 0$

$\Rightarrow u(x)$, $w_0(x)$

$$\boxed{\begin{aligned}
&A_{11}\frac{\partial^2 u_0}{\partial x^2} + A_{16}\frac{\partial^2 v_0}{\partial x^2} - B_{11}\frac{\partial^3 w_0}{\partial x^3} - \frac{\partial N_{xx}^T}{\partial x} = I_0\frac{\partial^2 u_0}{\partial t^2} - I_1\frac{\partial^3 w_0}{\partial x \partial t^2} \\
&A_{16}\frac{\partial^2 u_0}{\partial x^2} + A_{66}\frac{\partial^2 v_0}{\partial x^2} - B_{16}\frac{\partial^3 w_0}{\partial x^3} - \frac{\partial N_{xy}^T}{\partial x} = I_0\frac{\partial^2 v_0}{\partial t^2} \\
&B_{11}\frac{\partial^3 u_0}{\partial x^3} + B_{16}\frac{\partial^3 v_0}{\partial x^3} - D_{11}\frac{\partial^4 w_0}{\partial x^4} + \frac{\partial}{\partial x}\left(\hat{N}_{xx}\frac{\partial w_0}{\partial x}\right) - \frac{\partial^2 M_{xx}^T}{\partial x^2} + q \\
&\quad = I_0\frac{\partial^2 w_0}{\partial t^2} - I_2\frac{\partial^4 w_0}{\partial x^2 \partial t^2} + I_1\frac{\partial^3 u_0}{\partial x \partial t^2}
\end{aligned}}$$

Governing equations of cylindrical bending using CLPT (7.4-1 a,b,c)

where

$$(I_0, I_1, I_2) = \sum_{k=1}^{L}\int_{z_k}^{z_{k+1}} (1, z, z^2)\rho_0^{(k)}\, dz \qquad (7.4\text{-}2)$$

For a general lamination scheme, the three equations are fully coupled. In the case of cross-ply laminates, the second equation becomes uncoupled from the rest ($(\ )_{16}=0$).

Eq. (7.4-1) can be expressed in an alternative form:

$$A\frac{\partial^2 u_0}{\partial x^2} = B\frac{\partial^3 w_0}{\partial x^3} + A_{66}\frac{\partial N_{xx}^T}{\partial x} - A_{16}\frac{\partial N_{xy}^T}{\partial x} + A_{66}I_0\frac{\partial^2 u_0}{\partial t^2} - A_{16}I_0\frac{\partial^2 v_0}{\partial t^2} - A_{66}I_1\frac{\partial^3 w_0}{\partial x\partial t^2}$$

$$A\frac{\partial^2 v_0}{\partial x^2} = C\frac{\partial^3 w_0}{\partial x^3} + A_{11}\frac{\partial N_{xy}^T}{\partial x} - A_{16}\frac{\partial N_{xx}^T}{\partial x} + A_{11}I_0\frac{\partial^2 v_0}{\partial t^2} - A_{16}I_0\frac{\partial^2 u_0}{\partial t^2} + A_{16}I_1\frac{\partial^3 w_0}{\partial x\partial t^2}$$

$$D\frac{\partial^4 w_0}{\partial x^4} = \bar{B}\frac{\partial^2 N_{xx}^T}{\partial x^2} + \bar{C}\frac{\partial^2 N_{xy}^T}{\partial x^2} - (I_1 - \bar{B}I_0)\frac{\partial^3 u_0}{\partial x\partial t^2} + \bar{C}I_0\frac{\partial^3 v_0}{\partial x\partial t^2} - I_0\frac{\partial^2 w_0}{\partial t^2} + (I_2 - \bar{B}I_1)\frac{\partial^4 w_0}{\partial x^2\partial t^2} - \frac{\partial^2 M_{xx}^T}{\partial x^2} + \frac{\partial}{\partial x}\left(\hat{N}_{xx}\frac{\partial w_0}{\partial x}\right) + q$$

(7.4-3 a, b, c)

where

$$A = A_{11}A_{66} - A_{16}A_{16}, \quad B = B_{11}A_{66} - B_{16}A_{16}, \quad C = A_{11}B_{16} - A_{16}B_{11}$$

$$D = D_{11} - B_{11}\bar{B} - B_{16}\bar{C}, \quad \bar{B} = \frac{B}{A}, \quad \bar{C} = \frac{C}{A}$$

(7.4-4)

Note that $C=0$ for a cross-ply laminate ($A_{16}=B_{16}=D_{16}=0$) and v is identically zero unless N_{xy}^T is at least a linear function of x.