Composites

Lesson 22

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Example: Simply supported beam.

$$W(0)=0,\quad W(a)=0,\quad \frac{d^2W}{dx^2}(0)=0,\quad \frac{d^2W}{dx^2}(a)=0 \Rightarrow \begin{array}{c} c_2=c_3=c_4=0 \\ c_1 \sin\lambda a = 0, \text{ which implies } \lambda=\dfrac{n\pi}{a} \end{array}$$

(7.2-34) $\Rightarrow$

$$\omega_n=\left(\frac{n\pi}{a}\right)^2 a_0\sqrt{1+\frac{b\hat{N}_{xx}}{(\frac{n\pi}{a})^2E^b_{xx}I_{yy}}}\sqrt{\frac{1}{1+(\frac{n\pi}{a})^2\frac{\hat{I}_2}{\hat{I}_0}}}$$

if $\hat{I}_2=0 \Rightarrow$

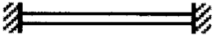
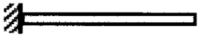
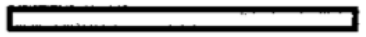
$$\omega_n=\left(\frac{n\pi}{a}\right)^2 a_0\sqrt{1+\frac{b\hat{N}_{xx}}{(\frac{n\pi}{a})^2E^b_{xx}I_{yy}}} \qquad (*)$$

Thus the effect of the axial tensile force $\hat{N}_{xx}$ is to increase the natural frequencies. If we have a very flexible beam, say a cable under large tension, the second term under the radical in Eq. $(*)$ becomes very large in comparision

With unit J; if n is not larg, we have

$$\omega_n \approx \frac{n\pi}{a}\sqrt{\frac{\hat{N}_{11}}{\hat{I}_0}} \qquad \text{(laminated cable)}$$

Table 4.2.3: Values of the constants and eigenvalues for natural vibration of laminated composite beams with various boundary conditions ($\lambda_n^4 \equiv \omega_n^2 I_0 / E_{xx}^b I_{yy} = (e_n/a)^4$). The classical laminate theory *without* rotary inertia is used.

End conditions at $x = 0$ and $x = a$	Constants[†]	Characteristic equation and values of $e_n \equiv \lambda_n a$
• Hinged-Hinged 	$c_1 \neq 0,\ c_2 = c_3 = c_4 = 0$	$\sin e_n = 0$ $e_n = n\pi$
• Fixed-Fixed 	$c_1 = -c_3 = 1/(\sin e_n - \sinh e_n)$ $-c_2 = c_4 = 1/(\cos e_n - \cosh e_n)$	$\cos e_n \cosh e_n - 1 = 0$ $e_n = 4.730, 7.853, \cdots$
• Fixed-Free 	$c_1 = -c_3 = 1/(\sin e_n + \sinh e_n)$ $-c_2 = c_4 = 1/(\cos e_n + \cosh e_n)$	$\cos e_n \cosh e_n + 1 = 0$ $e_n = 1.875, 4.694, \cdots$
• Free-Free 	$c_1 = c_3 = 1/(\sin e_n - \sinh e_n)$ $c_2 = c_4 = -1/(\cos e_n - \cosh e_n)$	$\cos e_n \cosh e_n - 1 = 0$ $e_n = 4.730, 7.853, \cdots$
• Hinged-Fixed 	$c_1 = 1/\sin e_n,\ c_3 = 1/\sinh e_n$ $c_2 = c_4 = 0$	$\tan e_n = \tanh e_n$ $e_n = 3.927, 7.069, \cdots$
• Hinged-Free 	$c_1 = 1/\sin e_n,\ c_3 = -1/\sinh e_n$ $c_2 = c_4 = 0$	$\tan e_n = \tanh e_n$ $e_n = 3.927, 7.069, \cdots$

† See Eq. (4.2.46a): $W(x) = c_1 \sin \lambda x + c_2 \cos \lambda x + c_3 \sinh \mu x + c_4 \cosh \mu x$.

Table 4.2.4: Maximum transverse deflections, critical buckling loads, and fundamental frequencies of laminated beams according to the classical beam theory ($E_1/E_2 = 25$, $G_{12} = G_{13} = 0.5E_2$, $G_{23} = 0.2E_2$, $\nu_{12} = 0.25$).

Laminate	Hinged-Hinged			Clamped-Clamped			Clamped-Free		
	$\bar{w}$	$\bar{N}$	$\bar{\omega}$	$\bar{w}$	$\bar{N}$	$\bar{\omega}$	$\bar{w}$	$\bar{N}$	$\bar{\omega}$
0	1.000 0.625	20.562	14.246 14.245 14.187	0.250 0.125	82.247	32.292 32.291 32.129	16.000 6.000	5.140	5.074 5.074 5.071
90	25.000 15.625	0.822	2.849	6.250 3.125	3.290	6.458	400.00 150.00	0.205	1.015
$(0/90)_s$	1.134 0.709	18.127	13.375	0.283 0.142	72.507	30.320	18.149 6.806	4.532	4.764
$(90/0)_s$	6.239 3.899	3.296	5.703	1.560 0.780	13.183	12.929	99.821 37.433	0.824	2.032
$(45/-45)_s$	14.308 8.942	1.437	3.766	3.577 1.788	5.748	8.537	228.93 85.847	0.359	1.341
Laminate A	1.607 1.005	12.790	11.236	0.402 0.201	51.162	25.469	25.721 9.645	3.197	4.002
Laminate B	2.801 1.751	7.341	8.512	0.700 0.350	29.366	19.296	44.813 16.805	1.835	3.032
Laminate C	7.945 4.966	2.588	5.054	1.986 0.993	10.351	11.456	127.13 47.673	0.647	1.800

Laminate $A = (0/\pm 45/90)_s$, Laminate $B = (45/0/-45/90)_s$, Laminate $C = (90/\pm 45/0)_s$.

Seval numerical examples

7.3. Analysis of Laminated Beam Using FSDT

7.3.1. Governing Equations

When applied to beams, FSDT is known as the Timoshenko beam theory.

The laminate constitutive equations for symmetric laminates, in the absence of in-plane forces, are given by:

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = [D] \begin{Bmatrix} \dfrac{\partial \phi_x}{\partial x} \\ \dfrac{\partial \phi_y}{\partial y} \\ \dfrac{\partial \phi_x}{\partial y} + \dfrac{\partial \phi_y}{\partial x} \end{Bmatrix} \qquad (7.3\text{-}1)$$

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = K \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial w_0}{\partial y} + \phi_y \\ \dfrac{\partial w_0}{\partial x} + \phi_x \end{Bmatrix} \qquad (7.3\text{-}2)$$

$$\begin{Bmatrix} \dfrac{\partial \phi_x}{\partial x} \\ \dfrac{\partial \phi_y}{\partial y} \\ \dfrac{\partial \phi_x}{\partial y} + \dfrac{\partial \phi_y}{\partial x} \end{Bmatrix} = [D]^{*} \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} \qquad (7.3\text{-}3)$$

$$\begin{Bmatrix} \dfrac{\partial w_0}{\partial y} + \phi_y \\ \dfrac{\partial w_0}{\partial x} + \phi_x \end{Bmatrix} = \frac{1}{K} \begin{bmatrix} A_{44}^{*} & A_{45}^{*} \\ A_{45}^{*} & A_{55}^{*} \end{bmatrix} \begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} \qquad (7.3\text{-}4)$$

where K is the shear corection coefficient, D_{ij}^{*} denote the elements of the inverse of $[D]$, and A_{ij}^{*} denote elements of the inverse of $[A]$.

As in previous section, we assume that

$M_y = M_{xy} = Q_y = \phi_y = 0$ and both w_0 and ϕ_x are functions of only x and t:

$$w_0 = w_0(x,t), \quad \phi_x = \phi_x(x,t) \qquad (7.3\text{-}5)$$

$$\Rightarrow \quad u(x,z) = z\,\phi_x(x,t), \quad w(x,z) = w_0(x) \qquad (7.3\text{-}6)$$

$$\longrightarrow \quad \varepsilon_x = z\,\frac{\partial \phi_x}{\partial x}, \quad 2\varepsilon_{xz} = \frac{\partial w_0}{\partial x} + \phi_x \qquad (7.3\text{-}7)$$

$$\longrightarrow \begin{cases} \dfrac{\partial \phi_x}{\partial x} = D_{11}^{*} M_x \\[2ex] \dfrac{\partial w_0}{\partial x} + \phi_x = \dfrac{A_{55}^{*}}{K} Q_x \end{cases} \qquad (7.3\text{-}8)$$

or

$$\begin{cases} E_x^b I_y \dfrac{\partial \phi_x}{\partial x} = M(x) & , M(x) = b\,M_x \,, \; E_x^b = \dfrac{12}{D_{11}^* h^3} \\[2ex] K\, G_{xz}^b\, bh \left(\dfrac{\partial w_0}{\partial x} + \phi_x\right) = Q(x) & , Q(x) = b\,Q_x \,, \; G_{xz}^b = \dfrac{1}{A_{55}^* h} \end{cases}$$

(7.3-9 a, b)

The equations of motion from Eq. (6.2-17) are:

$$\frac{\partial Q_x}{\partial x} + \hat{N}_x \frac{\partial^2 w_0}{\partial x^2} + q = I_0 \frac{\partial^2 w_0}{\partial t^2} \qquad (7.3\text{-}10a)$$

$$\frac{\partial M_x}{\partial x} - Q_x = I_2 \frac{\partial^2 \phi_x}{\partial t^2} \qquad (7.3\text{-}10b)$$

Using Eq. (7.3-9) in Eq. (7.3-10), the equations of motion can be recast in terms of the displacement functions:

$$\boxed{\begin{aligned} KG^b_{xz}bh\left(\frac{\partial^2 w_0}{\partial x^2}+\frac{\partial \phi_x}{\partial x}\right)+b\hat{N}_{xx}\frac{\partial^2 w_0}{\partial x^2}+\hat{q}=\hat{I}_0\frac{\partial^2 w_0}{\partial t^2} \\ E^b_{xx}I_{yy}\frac{\partial^2 \phi_x}{\partial x^2}-KG^b_{xz}bh\left(\frac{\partial w_0}{\partial x}+\phi_x\right)=\hat{I}_2\frac{\partial^2 \phi_x}{\partial t^2}\end{aligned}}$$

Governing equations of FSDT beam

(7.3-11 a,b)

where

$\hat{q}=bq$, $\hat{I}_0=bI_0$, $\hat{I}_2=bI_2$ (7.3-11 c)

$\rho\int z\,dA$ $\rho\int z^2 dA$

6.3-2 Bending

Bending of FSDT beam

$$KG^b_{xz}bh\left(\frac{d^2 w_0}{dx^2}+\frac{d\phi_x}{dx}\right)+\hat{q}=0 \qquad (7.3\text{-}12a)$$

$$E^b_{xx}I_{yy}\frac{d^2 \phi_x}{dx^2}-KG^b_{xz}bh\left(\frac{dw_0}{dx}+\phi_x\right)=0 \qquad (7.3\text{-}12b)$$

$$\Rightarrow E^b_{xx}I_{yy}\phi_x(x)=-\int_0^x\int_0^\zeta\int_0^\eta \hat{q}(\xi)\,d\xi\, d\eta\, d\zeta+c_1\frac{x^2}{2}+c_2 x+c_3 \qquad (7.3\text{-}13)$$

and

$$w_0(x) = -\frac{1}{E^b_{xx}I_{yy}}\left[-\int_0^x\int_0^\xi\int_0^\eta\int_0^\mu \hat{q}(\zeta)d\zeta d\mu d\eta d\xi + c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3x + c_4\right] \quad (7.3\text{-}14)$$

$$+ \frac{1}{KG^b_{xz}bh}\left[-\int_0^x\int_0^\xi \hat{q}(\zeta)d\zeta d\xi + c_1x\right] \quad (4.3.13\text{b}$$

It is informative to note from Eq. (7.3-14) that the transverse deflection of the Timoshenko beam theory consits of two parts, one due to pure bending and the other due to transverse shear:

$$w_0(x) = w_0^b(x) + w_0^s(x) \quad (7.3\text{-}15)$$

where

$$w_0^b(x) = \frac{1}{E^b_{xx}I_{yy}}\left[\int_0^x\int_0^\xi\int_0^\eta\int_0^\mu \hat{q}(\zeta)d\zeta d\mu d\eta d\xi - c_1\frac{x^3}{6} - c_2\frac{x^2}{2} - c_3x - c_4\right] \quad (7.3\text{-}16)$$

$$w_0^s(x) = \frac{1}{KG^b_{xz}bh}\left[-\int_0^x\int_0^\xi \hat{q}(\zeta)d\zeta d\xi + c_1x\right] \quad (4.3.$$

The pure bending deflection $w_0^b(x)$ is the same as that derived in the classical beam theory (CLPT). When the transverse shear stiffness is infinite, the shear deflection $w_0^s(x)$ goes to zero, and the Timoshenko beam theory solutions reduce to those of the classical beam theory (CLPT).

The expressions for in-plane stresses of the Timoshenko beam theory remain the same as those in the classical beam theory (see Eq. (7.2-16)).

$$\begin{aligned}
\sigma_{xx}^{(k)}(x,z) &= \frac{M(x)z}{b}\left(\bar{Q}_{11}^{(k)}D_{11}^{*}+\bar{Q}_{12}^{(k)}D_{12}^{*}+\bar{Q}_{16}^{(k)}D_{16}^{*}\right)\\
\sigma_{yy}^{(k)}(x,z) &= \frac{M(x)z}{b}\left(\bar{Q}_{12}^{(k)}D_{11}^{*}+\bar{Q}_{22}^{(k)}D_{12}^{*}+\bar{Q}_{26}^{(k)}D_{16}^{*}\right)\\
\sigma_{xy}^{(k)}(x,z) &= \frac{M(x)z}{b}\left(\bar{Q}_{16}^{(k)}D_{11}^{*}+\bar{Q}_{26}^{(k)}D_{12}^{*}+\bar{Q}_{66}^{(k)}D_{16}^{*}\right)
\end{aligned} \qquad (7.3\text{-}17)$$

And transverse shear stresses are the same as (7.2-19)

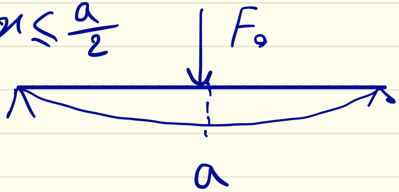
$$\sigma_{xz}^{(k)}(x,z) = -Q_x(x)\left(\bar{Q}_{11}^{(k)}D_{11}^{*} + \bar{Q}_{12}^{(k)}D_{12}^{*} + \bar{Q}_{16}^{(k)}D_{16}^{*}\right)\left(\frac{z^2 - z_k^2}{2}\right) + G^{(k)} \quad (4.2.15a)$$

(7.3-18)

$$\sigma_{zz}^{(k)}(x,z) = -\frac{dQ_x}{dx}\left(\bar{Q}_{11}^{(k)}D_{11}^{*} + \bar{Q}_{12}^{(k)}D_{12}^{*} + \bar{Q}_{16}^{(k)}D_{16}^{*}\right)\left(\frac{z^3 - z_k^3}{6}\right) + H^{(k)} \quad (4.2.15b)$$

Example: Simply supported beam.

$$M(x) = \frac{F_0 b x}{2}, \quad Q(x) = \frac{dM}{dx} = \frac{F_0 b}{2}, \quad 0 \leq x \leq \frac{a}{2}$$



From (7.3-9a) we have

$$\phi_x(x) = \frac{F_0 b}{4E_x^b I_y} x^2 + C_1$$

That's not the slope

$$u = u_0 + z\phi_x \xrightarrow{\text{symmetry}} \phi_x\left(\frac{a}{2}\right) = 0 \longrightarrow C_1 = -\frac{F_0 b a^2}{16E_x^b I_y}$$

$$\phi_x(x) = -\frac{F_0 b a^2}{16E_{xx}^b I_{yy}}\left[1 - 4\left(\frac{x}{a}\right)^2\right], \quad 0 \leq x \leq \frac{a}{2}$$

(I)

It is interesting to note from Eq. (I) that the rotation function $\phi_x(x)$ is the same as the slope $-dw_0/dx$ from the Euler–Bernoulli beam theory (i.e., ϕ_x is independent of transverse shear stiffness). Consequently, the bending moment and therefore the axial stress, is independent of shear deformation. In fact, ϕ_x is independent of shear deformation for all statically determinate beams and indeterminate beams with symmetric boundary conditions and loading (see Wang [27]). However, for general statically indeterminate beams, the rotation ϕ_x will depend on the shear stiffness $KG^b_{xz}bh$ (see Problem 4.11).

substituting for ϕ_x into Eq. (7.3-9b) we obtain

$$\frac{dw_0}{dx} = \frac{F_0 b a^2}{16 E_x^b I_y}\left[1 - 4\left(\frac{x}{a}\right)^2\right] + \frac{F_0 b}{2 K G^b_{xz} bh} \qquad (\text{II})$$