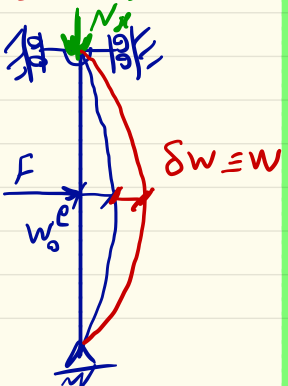
Composites — Lesson 21 — بسم الله الرحمن الرحيم

7.2-3 Buckling

In a beam subjected to axial compressive load $\hat{N}_x = -N_x^0$, if the small aditional disturbance results in a large response and the beam does not return to its original equilibrium configuration, the beam is said to be unstable. The onset of instability is called buckling

$$(7.2\text{-}8) \Rightarrow \boxed{\frac{d^4 w}{dx^4} + \frac{b N_x^0}{E_x^b I_y}\frac{d^2 w}{dx^2} = 0} \quad (7.2\text{-}21)$$

Buckling Eq. for CLPT beam



$w_0 = w_0^e + w$ — w: buckling deflection; w_0^e: prebuckling

w_0 and w_0^e satisfy Eq. (7.2-21) too.

$$E_x^b I_y \frac{d^4 w_0}{dx^4} + b N_x^0 \frac{d^2 w_0}{dx^2} = 0 \qquad (7.2\text{-}22)$$

$$E_x^b I_y \frac{d^4 w_0^e}{dx^4} + b N_x^0 \frac{d^2 w_0^e}{dx^2} = 0 \qquad (7.2\text{-}23)$$

By integrating (7.2-21) we have.

$$\boxed{\frac{d^2 w}{dx^2} + \frac{b N_x^0}{E_x^b I_y} w = K_1 x + K_2} \qquad (7.2\text{-}24)$$

General Solution for this equation is: (7.2-25)

$$w(x) = C_1 \sin(\lambda_b x) + C_2 \cos(\lambda_b x) + C_3 x + C_4$$

where

$$\lambda_b^2 = \frac{b N_x^0}{E_x^b I_y}, \quad C_3 = \frac{K_1}{\lambda_b^2}, \quad C_4 = \frac{K_2}{\lambda_b^2} \qquad (7.2\text{-}26)$$

The constants c_1, c_2, c_3 and c_4 can be determined using the boundary conditions of the beam.

we are interested in determining the values of λ_b for which there exists a nonzero solution $w(x)$, when beam experiences deflection. once such a λ_b is known (often there will be many), the buckling load is determined from Eq. (7.2-26)

$$N_x^0 = \left(\frac{E_x^b I_y}{b}\right) \lambda_b^2 \qquad (7.2\text{-}27)$$

The smallest value of N_x^0 which is given by the smallest value of λ_b is the critical buckling load. The buckling shape (or mode) is given by $w(x)$.

Example: Simply supported beam.

$$w_0(0)=0\ ,\ w_0(a)=0,\ M_x(0)=0\ ,\ M_x(a)=0$$

These boundary conditions imply

$$w(0)=0\ ,\ w(a)=0,\ \frac{d^2w}{dx^2}(0)=0,\ \frac{d^2w}{dx^2}(a)=0$$

$w(0)=0$: $\quad C_2+C_4=0$

$w''(0)=0$: $\quad -C_2\lambda_b^2=0$ which implies $C_2=0$, $C_4=0$

$w(a)=0$: $\quad c_1 \sin(\lambda_b a)+c_3 a=0$

$w''(a)=0$: $\quad c_1 \sin(\lambda_b a)=0$ which implies $c_3=0$

For a nontrivial solution, the condition

$$c_1 \sin(\lambda_b a)=0 \text{ implies that } \lambda_b a = n\pi\ ,\ n=1,2,\ldots$$

and the buckling load is given by

$$b N_x^0 = E_x^b I_y \left(\frac{n\pi}{a}\right)^2$$

$$\longrightarrow \quad w(x) = c_1 \sin\frac{n\pi x}{a} \quad , \; c_1 \neq 0$$

The critical buckling load becomes $(n=1)$

$$N_{cr} = \left(\frac{\pi}{a}\right)^2 \frac{E_x^b I_y}{b} = \frac{\pi^2}{12} \frac{E_x^b h^3}{a^2}$$

and the buckling mode (eigenfunction) associated with it is

$$w(x) = c_1 \sin\frac{\pi x}{a}$$

Example: clamped beam

$$w_0(0)=0, \quad \frac{dw_0}{dx}(0)=0, \quad w_0(a)=0, \quad \frac{dw_0}{dx}(a)=0$$

which can be expressed as

$$W(0)=0, \quad \frac{dW}{dx}(0)=0, \quad W(a)=0, \quad \frac{dW}{dx}(a)=0$$

we have

$W(0)=0$: $\quad C_2+C_4=0$

$W'(0)=0$: $\quad C_1\lambda_b+C_3=0$

$W(a)=0$: $\quad C_1\sin(\lambda_b a)+C_2\cos(\lambda_b a)+C_3 a+C_4=0$

$W'(a)=0$: $\quad C_1\lambda_b\cos(\lambda_b a)-C_2\lambda_b\sin(\lambda_b a)+C_3=0$

Expressing these equations in terms of constants c_1 and c_2 we obtain

$$c_1(\sin(\lambda_b a) - \lambda_b a) + c_2(\cos\lambda_b a - 1) = 0$$

$$c_1(\cos(\lambda_b a) - 1) - c_2 \sin(\lambda_b a) = 0$$

For a nontrivial solution, the determinate of the coefficient matrix of the above two equations must be zero (eigenvalue problem)

$$\begin{vmatrix} \sin(\lambda_b a) - \lambda_b a & \cos(\lambda_b a) - 1 \\ \cos(\lambda_b a) - 1 & -\sin(\lambda_b a) \end{vmatrix} = 0 \qquad (*)$$

$$\rightarrow \lambda_b a \sin(\lambda_b a) + 2\cos(\lambda_b a) - 2 = 0$$

characteristic equation

The solution of equation $(*)$, known as the characteristic equation, gives the eigenvalues $e_n \equiv \lambda_b a$, and the buckling load is calculated from Eq. (7.2-27).

A plot of the function $f(e_n) = e_n \sin(e_n) + 2\cos(e_n) - 2$ against e_n shows that $f(e_n)$ is zero at

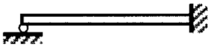
$$e_n = 0,\ 6.2832\,(=2\pi),\ 8.9868,\ 12.5664\,(=4\pi), \ldots \ (\lambda_{2n-1}a = 2n\pi).$$

Hence, the critical (smallest) buckling load is

$$N_{cr} = \left(\frac{e_n}{a}\right)^2 \left(\frac{E_x^b I_y}{b}\right) = \left(\frac{2\pi}{a}\right)^2 \left(\frac{E_x^b I_y}{b}\right)$$

$$= \left(\frac{\pi^2}{3}\right)\left(\frac{E_x^b h^3}{a^2}\right)$$

Table 4.2.2: Values of the constants and eigenvalues for buckling of laminated composite beams with various boundary conditions ($\lambda^2 \equiv bN_{xx}^0/E_{xx}^b I_{yy} = (e_n/a)^2$). The classical laminate theory is used.

End conditions at $x = 0$ and $x = a$	Constants†	Characteristic equation and values* of $e_n \equiv \lambda_n a$
• Hinged-Hinged 	$c_1 \neq 0,\ c_2 = c_3 = c_4 = 0$	$\sin e_n = 0$ $e_n = n\pi$
• Fixed-Fixed 	$c_1 = 1/(\sin e_n - e_n)$ $c_3 = -1/\lambda_n$ $c_2 = -c_4 = 1/(\cos e_n - 1)$	$e_n \sin e_n = 2(1 - \cos e_n)$ $e_n = 2\pi, 8.987, 4\pi, \cdots$
• Fixed-Free 	$c_1 = c_3 = 0$ $c_2 = -c_4 \neq 0$	$\cos e_n = 0$ $e_n = (2n-1)\pi/2$
• Free-Free 	$c_1 = c_3 = 0$ $c_2 \neq 0,\ c_4 \neq 0$	$\sin e_n = 0$ $e_n = n\pi$
• Hinged-Fixed 	$c_1 = 1/e_n \cos e_n,\ c_3 = -1$ $c_2 = c_4 = 0$	$\tan e_n = e_n$ $e_n = 4.493, 7.725, \cdots$

† See Eq. (4.2.28): $W(x) = c_1 \sin \lambda_b x + c_2 \cos \lambda_b x + c_3 x + c_4$.

*For critical buckling load, only the first (minimum) value of $e = \lambda a$ is needed.

7.2-4 Vibration

For natural vibration, the solution is assumed to be periodic

$$w_0(x,t) = W(x)\, e^{i\omega t}, \quad i = \sqrt{-1} \qquad (7.2-28)$$

In the absence of applied transverse load q, the governing equation (7.2-8) reduces to:

$$E_x^b I_y \frac{d^4 W}{dx^4} - b\hat{N}_x \frac{d^2 W}{dx^2} = \omega^2 \hat{I}_0 W - \omega^2 \hat{I}_2 \frac{d^2 W}{dx^2} \qquad (7.2-29)$$

Equation (7.2-29) has the general form

$$p \frac{d^4 W}{dx^4} + q \frac{d^2 W}{dx^2} - rW = 0 \qquad (7.2-30)$$

where

$$p = E_x^b I_y, \quad q = \omega^2 \hat{I}_2 - b\hat{N}_x, \quad r = \omega^2 \hat{I}_0 \qquad (7.2\text{-}31)$$

The general solution of Eq (7.2-30) is:

$$\boxed{w(x) = C_1 \sin(\lambda x) + C_2 \cos(\lambda x) + C_3 \sinh(\mu x) + C_4 \cosh(\mu x)} \quad (7.2\text{-}32a)$$

Vibration solution for CLPT beam

$$\lambda = \sqrt{\frac{1}{2p}\left(q + \sqrt{q^2 + 4pr}\right)}, \quad \mu = \sqrt{\frac{1}{2p}\left(-q + \sqrt{q^2 + 4pr}\right)} \quad (7.2\text{-}32b)$$

and C_1, C_2, C_3 and C_4 are constants, which are to be determined using the boundary conditions.

$$(7.2\text{-}32b) \Rightarrow \quad (2p\lambda^2 - q)^2 = q^2 + 4pr \quad \text{or} \quad p\lambda^4 - q\lambda^2 - r = 0$$

$$(2p\mu^2 + q) = q^2 + 4pr \quad \text{or} \quad p\mu^4 + q\mu^2 - r = 0$$

(7.2-33 a,b)

Substituting for p, q and r from Eq. (7.2-31) in Eq. (7.2-33) and solving for ω^2, we obtain

$$\omega^2 = \lambda^4 \left(\frac{E^b_{xx} I_{yy}}{\hat{I}_0}\right)\left(\frac{1+P_1}{1+R_1}\right), \quad P_1 = \frac{b\hat{N}_{xx}}{E^b_{xx} I_{yy} \lambda^2}, \quad R_1 = \frac{\hat{I}_2}{\hat{I}_0}\lambda^2 \qquad (7.2\text{-}34a)$$

$$\omega^2 = \mu^4 \left(\frac{E^b_{xx} I_{yy}}{\hat{I}_0}\right)\left(\frac{1-P_2}{1-R_2}\right), \quad P_2 = \frac{b\hat{N}_{xx}}{E^b_{xx} I_{yy} \mu^2}, \quad R_2 = \frac{\hat{I}_2}{\hat{I}_0}\mu^2 \qquad (7.2\text{-}34b)$$

when the applied axial load is zero, the frequency of vibration can be calculated from

$$\omega^2 = \lambda^4 \frac{E^b_{xx} I_{yy}}{\hat{I}_0}\left(1 - \frac{\hat{I}_2 \lambda^2}{\hat{I}_0 + \hat{I}_2 \lambda^2}\right) = \mu^4 \frac{E^b_{xx} I_{yy}}{\hat{I}_0}\left(1 + \frac{\hat{I}_2 \mu^2}{\hat{I}_0 - \hat{I}_2 \mu^2}\right) \qquad (7.2\text{-}35)$$

It is clear from the first expression that rotary inertia decreases the frequency of natural vibration. If the rotary inertia is neglected, we hav $\lambda \equiv \mu$ and

$$\omega = \lambda^2 a_o \,, \qquad a_o = \sqrt{\frac{E_x^b I_y}{\hat{I}_o}} \qquad (7.2.36)$$