Composites   Lesson 20   بسم الله الرحمن الرحيم

In order to cast Eq. (7.2-4) in the familiar form used in the classical Euler-Bernoulli beam theory, we introduce the quantities:

$$M = b M_{xx} \;,\quad Q = b Q_x \;,\quad E_x^b = \frac{12}{h^3 D_{11}^*} = \frac{b}{I_y D_{11}^*} \;,\quad I_y = \frac{bh^3}{12} \qquad (7.2\text{-}5)$$

$$\Rightarrow \quad \frac{\partial^2 w_0}{\partial x^2} = -\frac{M}{E^b I_y} \quad \text{or} \quad M(x) = -E I_y \frac{\partial^2 w_0}{\partial x^2} \qquad (7.2\text{-}6)$$

b: with , h: thickness

The equation of motion of laminated beams can be obtained directly from Eq. (6.1-21) by setting all terms involving differentiation with respect to y to zero (only the third one).

$$\frac{\partial^2 M_{xx}}{\partial x^2} + \hat{N}_x \frac{\partial^2 w_0}{\partial x^2} + q = I_0 \frac{\partial^2 w_0}{\partial t^2} - I_2 \frac{\partial^4 w_0}{\partial x^2 \partial t^2} \qquad (7.2\text{-}7)$$

for symmetrically laminated long beams, bending and extention of beam are independent to each others, thus we can use (7.2-4)

$$\xrightarrow[7.2\text{-}4]{7.2\text{-}7} \boxed{-\frac{\partial^2}{\partial x^2}\left(E_x^b I_y \frac{\partial^2 w_0}{\partial x^2}\right) + b\hat{N}_x \frac{\partial^2 w_0}{\partial x^2} + \hat{q} = \hat{I}_0 \frac{\partial^2 w_0}{\partial t^2} - \hat{I}_2 \frac{\partial^4 w_0}{\partial x^2 \partial t^2}} \qquad (7.2\text{-}8)$$

Governing Eq. of CLPT symmetric long beam

where $\hat{N}_x$ is the applied axial load, and

$$\hat{q} = bq, \quad \hat{I}_0 = bI_0, \quad \hat{I}_2 = bI_2, \quad I_i = b\int_{-h/2}^{h/2} \rho z^i \, dz \qquad i = 0, 1, 2$$

$$(7.2\text{-}9)$$

Boundary Conditions:

Geometric $\quad w_0, \; \dfrac{\partial w_0}{\partial x} \quad$ specify $\qquad (7.2\text{-}10)$

Force $\quad Q = \dfrac{\partial M}{\partial x}, \; M \quad$ specify

7.2.2 Bending

For static bending without the axial force, $\hat{N}_x = 0$, Eqs. (7.2-6) and (7.2-8) take the form:

$$\frac{d^2 w_0}{dx^2} = -\frac{M}{E_x^b I_y}, \quad E_x^b I_y \frac{d^4 w_0}{dx^4} = \hat{q} \qquad (7.2\text{-}11\ a,b)$$

where $\hat{q} = bq$. Equation (a) is the most convenient when it is possible to express the bending moment M in terms of the applied loads. For indeterminate beams, use of Eq. (b) is is more convenient.

General Solution

$$(7.2\text{-}11\ a) \longrightarrow E_x^b I_y w_0(x) = -\int_0^x \left[\int_0^\eta M(\xi) d\xi \right] d\eta + b_1 x + b_2 \qquad (7.2\text{-}12)$$

(7.2-11 b) $\longrightarrow$

$$E_{xx}^{b} I_{yy} w_0(x) = \int_0^x \left\{ \int_0^\xi \left[\int_0^\eta \left(\int_0^\zeta \hat{q}(\mu) d\mu \right) \, d\zeta \right] \, d\eta \right\} d\xi + c_1 \frac{x^3}{6} + c_2 \frac{x^2}{2} + c_3 x + c_4 \qquad (7.2\text{-}13)$$

calculation of stresses

$$\{\sigma\}_k = [\bar{Q}]_k \{\varepsilon\}$$

$$\{\varepsilon\} \hookrightarrow \{\varepsilon\}^{\circ} + z\{\varepsilon\}^{(1)}$$

$$\{\sigma\}_k = z[\bar{Q}]_k \begin{Bmatrix} -\dfrac{\partial^2 w_0}{\partial x^2} \\ -\dfrac{\partial^2 w_0}{\partial y^2} \\ -2\dfrac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix} \qquad (7.2\text{-}14)$$

on the other hand, from (7.2-21)

$$\{\sigma\}_k = \frac{z}{b}[\bar{Q}]_k [D^*] \begin{Bmatrix} m \\ 0 \\ 0 \end{Bmatrix} \qquad (7.2\text{-}15)$$

$$\left(m_n = \frac{m}{b}\right)$$

$$\sigma_{xx}^{(k)}(x,z) = \frac{M(x)z}{b}\left(\bar{Q}_{11}^{(k)}D_{11}^{*} + \bar{Q}_{12}^{(k)}D_{12}^{*} + \bar{Q}_{16}^{(k)}D_{16}^{*}\right)$$
$$\sigma_{yy}^{(k)}(x,z) = \frac{M(x)z}{b}\left(\bar{Q}_{12}^{(k)}D_{11}^{*} + \bar{Q}_{22}^{(k)}D_{12}^{*} + \bar{Q}_{26}^{(k)}D_{16}^{*}\right)$$
$$\sigma_{xy}^{(k)}(x,z) = \frac{M(x)z}{b}\left(\bar{Q}_{16}^{(k)}D_{11}^{*} + \bar{Q}_{26}^{(k)}D_{12}^{*} + \bar{Q}_{66}^{(k)}D_{16}^{*}\right)$$
(7.2-16)

In the classical beam theory, the interlaminar stresses (σ_{xz}, σ_{zz}) are identically zero when computed using the constitutive Eqs. However, these stresses do exist in reality, and they can be responsible for failures in composite laminates used. Interlaminar stresses may be computed using the equilibrium equations of 3-D elasticity:

$$0 = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z}$$
$$0 = \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z}$$
$$0 = \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}$$
(7.2-17)

$\Rightarrow$

$$\sigma_{xz}^{(k)} = -\int_{z_k}^{z} \left(\frac{\partial \sigma_{xx}^{(k)}}{\partial x} + \frac{\partial \sigma_{xy}^{(k)}}{\partial y} \right) dz + G^{(k)}$$
$$\sigma_{yz}^{(k)} = -\int_{z_k}^{z} \left(\frac{\partial \sigma_{xy}^{(k)}}{\partial x} + \frac{\partial \sigma_{yy}^{(k)}}{\partial y} \right) dz + F^{(k)}$$
$$\sigma_{zz}^{(k)} = -\int_{z_k}^{z} \left(\frac{\partial \sigma_{xz}^{(k)}}{\partial x} + \frac{\partial \sigma_{yz}^{(k)}}{\partial y} \right) dz + H^{(k)} \qquad (7.2\text{-}18)$$

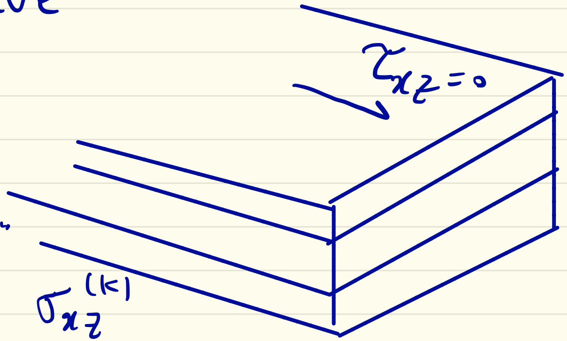
where $(\sigma_x^{(k)}, \sigma_y^{(k)}, \sigma_{xy}^{(k)})$ are known from Eq. (7.2-16) and $G^{(k)}$, $F^{(k)}$, and $H^{(k)}$ are constants.

For beams, all variables are independet of y and $v=0$. Hence, derivatives with respect to y are zero. For example from Eq. (7.2-18) we obtain:

$$\sigma_{xz}^{(k)}(x,z) = -Q_x(x)\left(\bar{Q}_{11}^{(k)} D_{11}^* + \bar{Q}_{12}^{(k)} D_{12}^* + \bar{Q}_{16}^{(k)} D_{16}^*\right)\left(\frac{z^2 - z_k^2}{2}\right) + G^{(k)} \qquad (7.2\text{-}19)$$
$$\sigma_{zz}^{(k)}(x,z) = -\frac{dQ_x}{dx}\left(\bar{Q}_{11}^{(k)} D_{11}^* + \bar{Q}_{12}^{(k)} D_{12}^* + \bar{Q}_{16}^{(k)} D_{16}^*\right)\left(\frac{z^3 - z_k^3}{6}\right) + H^{(k)}$$

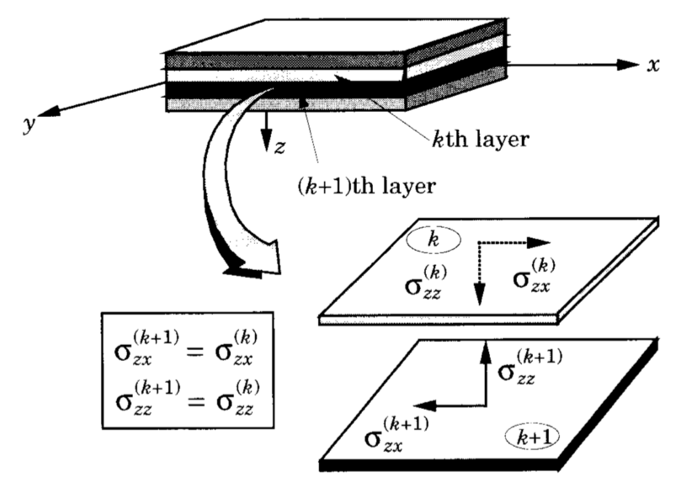
From the top and bottom surface boundary conditions we can find $G^{(1)}$ and $H^{(1)}$. Here we have

$$G^{(1)} = 0 \quad \text{and} \quad H^{(1)} = 0$$



Constants $G^{(k)}$ and $H^{(k)}$ for $k = 2, 3, \ldots$ are determined by requiring that $\sigma_{xz}^{(k)}$ and $\sigma_{zz}^{(k)}$ be continuous at the layer interfaces

$$\sigma_{xz}^{(k)}(x, z_{k+1}) = \sigma_{xz}^{(k+1)}(x, z_{k+1}), \quad \sigma_{zz}^{(k)}(x, z_{k+1}) = \sigma_{zz}^{(k+1)}(x, z_{k+1})$$



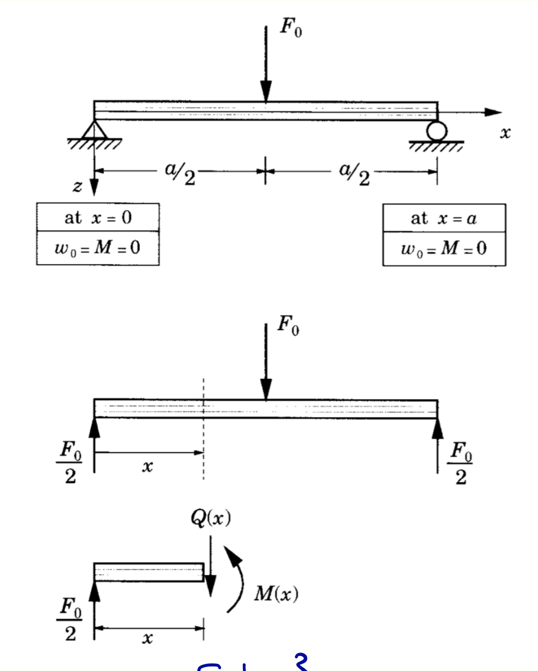
This gives, for $K=1,2,\ldots,$ the result

$$G^{(k+1)} = -Q_x(x)\left(\bar{Q}_{11}^{(k)}D_{11}^* + \bar{Q}_{12}^{(k)}D_{12}^* + \bar{Q}_{16}^{(k)}D_{16}^*\right)\left(\frac{z_{k+1}^2 - z_k^2}{2}\right) + G^{(k)}$$
$$= \sigma_{xz}^{(k)}(x, z_{k+1}) \quad (4.2$$

(7.2-20)

$$H^{(k+1)} = -\frac{dQ_x}{dx}\left(\bar{Q}_{11}^{(k)}D_{11}^* + \bar{Q}_{12}^{(k)}D_{12}^* + \bar{Q}_{16}^{(k)}D_{16}^*\right)\left(\frac{z_{k+1}^3 - z_k^3}{6}\right) + H^{(k)}$$
$$= \sigma_{zz}^{(k)}(x, z_{k+1}) \quad (4.2$$

Example: Three-point-bending



$$M(x) = \frac{(F_0 b)\, x}{2} \quad , \quad \text{for } 0 \le x \le \frac{a}{2}$$

(7.2-11 a) $\longrightarrow$

$$E_x^b I_y w_0(x) = -\frac{F_0 b x^3}{12} + C_1 x + C_2$$

c_1, c_2 can be evaluated using the boundary conditions:

$$w_0(0) = 0 \quad , \quad \left.\frac{dw_0}{dx}\right|_{a/2} = 0$$

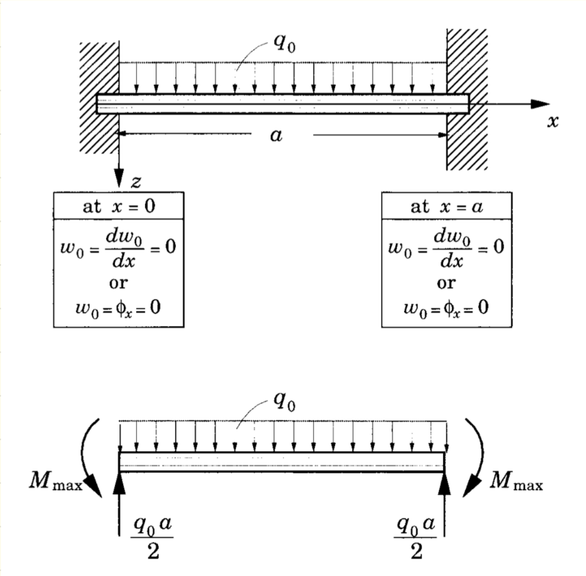
$$w_0(x) = \frac{F_0 b a^3}{48 E I_y}\left[3\left(\frac{x}{a}\right) - 4\left(\frac{x}{a}\right)^3\right]$$

The maximum in-plane stress σ_x occurs at $x = \frac{a}{2}$

(7.2-16)

$$\sigma_x^{(k)}\left(\frac{a}{2}, z\right) = \frac{F_o a z}{4}\left(\bar{Q}_{11}^{(k)} D_{11}^{*} + \bar{Q}_{12}^{(k)} D_{12}^{*} + \bar{Q}_{16}^{(k)} D_{16}^{*}\right)$$

Example:



By using (7.2-11 b) we have:

$$E I_y w_0(x) = \frac{q_0 b x^4}{24} + C_1 \frac{x^3}{6} + C_2 \frac{x^2}{2} + C_3 x + C_4$$

For the full beam case we have:

$$w_0(0) = 0,\quad w_0(a) = 0,\quad \left.\frac{dw_0}{dx}\right|_{0,a} = 0$$

For the half beam model we have

$$w_0(0) = 0,\quad \left.\frac{dw_0}{dx}\right|_{0} = 0,\quad \left.\frac{dw_0}{dx}\right|_{a/2} = 0$$

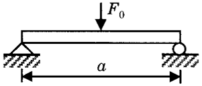
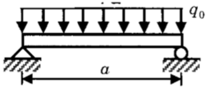
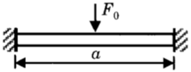
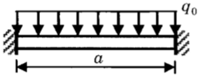
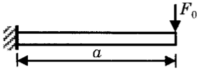
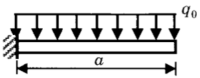
$$Q\left(\frac{a}{2}\right) = \frac{dM}{dx} = -E_x I_y \frac{d^3 w_0}{dx^3}\left(\frac{a}{2}\right) = 0$$

In the both cases we obtain:

$$w_0(x) = \frac{q_0 b a^4}{24 E I}\left[\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)\right]^2$$

$$M(x) = -\frac{q_0 b a^2}{12}\left[1-6\left(\frac{x}{a}\right)+a\left(\frac{x}{a}\right)^2\right], \quad M_{max} = -\frac{q_0 b a^2}{12}$$

$$\sigma_x^{(k)}(0,z) = -\frac{q_0 a^2 z}{12}\left(Q_{11}^{(k)} D_{11}^{*} + Q_{12}^{(k)} D_{12}^{*} + Q_{16}^{(k)} D_{16}^{*}\right)$$

Laminated Beam	Deflection, $w_0(x)$	w_{max} and M_{max}
• *Hinged-Hinged*		
Central point load 	$\frac{c_1}{48}\left[3\left(\frac{x}{a}\right)-4\left(\frac{x}{a}\right)^3\right]$	$w^c_{max}=\frac{1}{48}c_1$ $M^c_{max}=-\frac{1}{4}c_3$
Uniform load 	$\frac{c_2}{24}\left[\left(\frac{x}{a}\right)-2\left(\frac{x}{a}\right)^3+\left(\frac{x}{a}\right)^4\right]$	$w^c_{max}=\frac{5}{384}c_2$ $M^c_{max}=-\frac{1}{8}c_4$
• *Fixed-Fixed*		
Central point load 	$\frac{c_1}{48}\left[3\left(\frac{x}{a}\right)^2-4\left(\frac{x}{a}\right)^3\right]$	$w^c_{max}=\frac{1}{192}c_1$ $M^0_{max}=\frac{1}{8}c_3$
Uniform load 	$\frac{c_2}{24}\left[\left(\frac{x}{a}\right)^2-\left(\frac{x}{a}\right)\right]^2$	$w^c_{max}=\frac{1}{384}c_2$ $M^0_{max}=\frac{1}{12}c_4$
• *Fixed-Free*		
Point load at free end 	$\frac{c_1}{6}\left[3\left(\frac{x}{a}\right)^2-\left(\frac{x}{a}\right)^3\right]$	$w^a_{max}=\frac{1}{3}c_1$ $M^0_{max}=c_3$
Uniform load 	$\frac{c_2}{24}\left[6\left(\frac{x}{a}\right)^2-4\left(\frac{x}{a}\right)^3+\left(\frac{x}{a}\right)^4\right]$	$w^a_{max}=\frac{1}{8}c_2$ $M^0_{max}=\frac{1}{2}c_4$