Composites Lesson 19 بسم الله الرحمن الرحيم

From (6.2-8) by using integrate by parts to revile the virtual generalized displacements ($\delta u_o, \delta v_o, \delta w_o, \delta\phi_x, \delta\phi_y$) we obtain:

$$\begin{aligned}
0 = \int_0^T \int_{\Omega_0} \Big[ & -\left(N_{xx,x} + N_{xy,y} - I_0\ddot{u}_0 - I_1\ddot{\phi}_x\right)\delta u_0 \\
& -\left(N_{xy,x} + N_{yy,y} - I_0\ddot{v}_0 - I_1\ddot{\phi}_y\right)\delta v_0 \\
& -\left(M_{xx,x} + M_{xy,y} - Q_x - I_2\ddot{\phi}_x - I_1\ddot{u}_0\right)\delta\phi_x \\
& -\left(M_{xy,x} + M_{yy,y} - Q_y - I_2\ddot{\phi}_y - I_1\ddot{v}_0\right)\delta\phi_y \\
& -\left(Q_{x,x} + Q_{y,y} + \mathcal{N}(w_0) + q - I_0\ddot{w}_0\right)\delta w_0 \Big] dxdy \\
+ \int_0^T \int_{\Gamma} \Big[ & \left(N_{nn} - \hat{N}_{nn}\right)\delta u_n + \left(N_{ns} - \hat{N}_{ns}\right)\delta u_s + \left(Q_n - \hat{Q}_n\right)\delta w_0 \\
& + \left(M_{nn} - \hat{M}_{nn}\right)\delta\phi_n + \left(M_{ns} - \hat{M}_{ns}\right)\delta\phi_s \Big] dsdt
\end{aligned} \qquad (6.2\text{-}15)$$

where

$$\mathcal{N}(w_0) = \frac{\partial}{\partial x}\left(N_{xx}\frac{\partial w_0}{\partial x} + N_{xy}\frac{\partial w_0}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w_0}{\partial x} + N_{yy}\frac{\partial w_0}{\partial y}\right)$$

$$\mathcal{P}(w_0) = \left(N_{xx}\frac{\partial w_0}{\partial x} + N_{xy}\frac{\partial w_0}{\partial y}\right) n_x + \left(N_{xy}\frac{\partial w_0}{\partial x} + N_{yy}\frac{\partial w_0}{\partial y}\right) n_y$$

(6.2.16)

The Euler-lagrage equations are obtained by setting the coefficients of $\delta u_0$, $\delta v_0$, $\delta w_0$, and $\delta\phi_x$ and $\delta\phi_y$ in $\Omega_0$ to zero separately:

$$\boxed{\begin{aligned}
\delta u_0 &: \quad \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0\frac{\partial^2 u_0}{\partial t^2} + I_1\frac{\partial^2 \phi_x}{\partial t^2} \\
\delta v_0 &: \quad \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = I_0\frac{\partial^2 v_0}{\partial t^2} + I_1\frac{\partial^2 \phi_y}{\partial t^2} \\
\delta w_0 &: \quad \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \mathcal{N}(w_0) + q = I_0\frac{\partial^2 w_0}{\partial t^2} \\
\delta \phi_x &: \quad \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = I_2\frac{\partial^2 \phi_x}{\partial t^2} + I_1\frac{\partial^2 u_0}{\partial t^2} \\
\delta \phi_y &: \quad \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_y = I_2\frac{\partial^2 \phi_y}{\partial t^2} + I_1\frac{\partial^2 v_0}{\partial t^2}
\end{aligned}}$$

(6.2.17)

Equations of Motion for FSDT Plates

primary variables: $u_n, u_s, w_o, \phi_n, \phi_s$

(6.2-18)

secondary variables: $N_n, N_{ns}, Q_n, M_n, M_{ns}$

The natural b.c. are:

$$N_n - \hat{N}_n = 0, \quad N_{ns} - \hat{N}_{ns} = 0, \quad Q_n - \hat{Q}_n = 0$$

$$M_n - \hat{M}_n = 0, \quad M_{ns} - \hat{M}_{ns} = 0 \qquad (6.2\text{-}19)$$

where

$$Q_n = Q_x n_x + Q_y n_y + P(w_o)$$

# 6.2-3 Laminate Constitutive Equations

we had

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_{xy}\end{Bmatrix}_K = [\bar{Q}]_K \left( \begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix} - \begin{Bmatrix}\alpha_x\\ \alpha_y\\ \alpha_{xy}\end{Bmatrix}\Delta t \right) - \begin{bmatrix}0 & 0 & \bar{e}_{31}\\ 0 & 0 & \bar{e}_{32}\\ 0 & 0 & \bar{e}_{36}\end{bmatrix} \begin{Bmatrix}E_x\\ E_y\\ E_z\end{Bmatrix}_K \quad (6.2\text{-}20)$$

By integrating:

$$\begin{Bmatrix}N_x\\ N_y\\ N_{xy}\end{Bmatrix} = [A]\begin{Bmatrix}\varepsilon_x^0\\ \varepsilon_y^0\\ \gamma_{xy}^0\end{Bmatrix} + [B]\begin{Bmatrix}\varepsilon_x^{(1)}\\ \varepsilon_y^{(1)}\\ \gamma_{xy}^{(1)}\end{Bmatrix} \quad (6.2\text{-}21)$$

$$\begin{Bmatrix}M_x\\ M_y\\ M_{xy}\end{Bmatrix} = [B]\begin{Bmatrix}\varepsilon_x^0\\ \varepsilon_y^0\\ \gamma_{xy}^0\end{Bmatrix} + [D]\begin{Bmatrix}\varepsilon_x^{(1)}\\ \varepsilon_y^{(1)}\\ \gamma_{xy}^{(1)}\end{Bmatrix} \quad (6.2\text{-}22)$$

In FSDT The A, B and D stiffnes matrices don't change from CLPT ones.

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 \\ \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2 \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y} \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \end{Bmatrix}$$

(6.2.23)

Constitutive Eqs. fo FSDT

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 \\ \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2 \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y} \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \end{Bmatrix}$$

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = K \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} \frac{\partial w_0}{\partial y} + \phi_y \\ \frac{\partial w_0}{\partial x} + \phi_x \end{Bmatrix}$$

(*)

## 6.2-4 Equations of Motion in Terms of Displacements

The equations of motion (6.2-17) can be expressed in terms of displacements $(u_0, v_0, w_0, \phi_x, \phi_y)$ by substituting for the force and moment resultants from Eqs. (6.2-23)

$$
\begin{aligned}
&A_{11}\left(\frac{\partial^2 u_0}{\partial x^2}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x^2}\right)+A_{12}\left(\frac{\partial^2 v_0}{\partial y\partial x}+\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial y\partial x}\right)+\\
&A_{16}\left(\frac{\partial^2 u_0}{\partial y\partial x}+\frac{\partial^2 v_0}{\partial x^2}+\frac{\partial^2 w_0}{\partial x^2}\frac{\partial w_0}{\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial y\partial x}\right)+\\
&B_{11}\frac{\partial^2\phi_x}{\partial x^2}+B_{12}\frac{\partial^2\phi_y}{\partial y\partial x}+B_{16}\left(\frac{\partial^2\phi_x}{\partial x\partial y}+\frac{\partial^2\phi_y}{\partial x^2}\right)+\\
&A_{16}\left(\frac{\partial^2 u_0}{\partial x\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x\partial y}\right)+A_{26}\left(\frac{\partial^2 v_0}{\partial y^2}+\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial y^2}\right)+\\
&A_{66}\left(\frac{\partial^2 u_0}{\partial y^2}+\frac{\partial^2 v_0}{\partial x\partial y}+\frac{\partial^2 w_0}{\partial x\partial y}\frac{\partial w_0}{\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial y^2}\right)+\\
&B_{16}\frac{\partial^2\phi_x}{\partial x\partial y}+B_{26}\frac{\partial^2\phi_y}{\partial y^2}+B_{66}\left(\frac{\partial^2\phi_x}{\partial y^2}+\frac{\partial^2\phi_y}{\partial y\partial x}\right)-\\
&\left(\frac{\partial N^T_{xx}}{\partial x}+\frac{\partial N^T_{xy}}{\partial y}\right)-\left(\frac{\partial N^P_{xx}}{\partial x}+\frac{\partial N^P_{xy}}{\partial y}\right)=I_0\frac{\partial^2 u_0}{\partial t^2}+I_1\frac{\partial^2\phi_x}{\partial t^2}
\end{aligned}
\qquad (6.2\text{-}24a)
$$

$$
\begin{aligned}
&A_{16}\left(\frac{\partial^2 u_0}{\partial x^2}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x^2}\right)+A_{26}\left(\frac{\partial^2 v_0}{\partial y \partial x}+\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial y \partial x}\right)+\\
&A_{66}\left(\frac{\partial^2 u_0}{\partial y \partial x}+\frac{\partial^2 v_0}{\partial x^2}+\frac{\partial^2 w_0}{\partial x^2}\frac{\partial w_0}{\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial y \partial x}\right)+\\
&B_{16}\frac{\partial^2 \phi_x}{\partial x^2}+B_{26}\frac{\partial^2 \phi_y}{\partial y \partial x}+B_{66}\left(\frac{\partial^2 \phi_x}{\partial x \partial y}+\frac{\partial^2 \phi_y}{\partial x^2}\right)+\\
&A_{12}\left(\frac{\partial^2 u_0}{\partial x \partial y}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x \partial y}\right)+A_{22}\left(\frac{\partial^2 v_0}{\partial y^2}+\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial y^2}\right)+\\
&A_{26}\left(\frac{\partial^2 u_0}{\partial y^2}+\frac{\partial^2 v_0}{\partial x \partial y}+\frac{\partial^2 w_0}{\partial x \partial y}\frac{\partial w_0}{\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial y^2}\right)+\\
&B_{12}\frac{\partial^2 \phi_x}{\partial x \partial y}+B_{22}\frac{\partial^2 \phi_y}{\partial y^2}+B_{26}\left(\frac{\partial^2 \phi_x}{\partial y^2}+\frac{\partial^2 \phi_y}{\partial x \partial y}\right)-\\
&\left(\frac{\partial N_{xy}^T}{\partial x}+\frac{\partial N_{yy}^T}{\partial y}\right)-\left(\frac{\partial N_{xy}^P}{\partial x}+\frac{\partial N_{yy}^P}{\partial y}\right)=I_0\frac{\partial^2 v_0}{\partial t^2}+I_1\frac{\partial^2 \phi_y}{\partial t^2}
\end{aligned}
$$

(6.2-24b)

$$
\begin{aligned}
&KA_{55}\left(\frac{\partial^2 w_0}{\partial x^2}+\frac{\partial \phi_x}{\partial x}\right)+KA_{45}\left(\frac{\partial^2 w_0}{\partial y \partial x}+\frac{\partial \phi_y}{\partial x}\right)+\\
&KA_{45}\left(\frac{\partial^2 w_0}{\partial x \partial y}+\frac{\partial \phi_x}{\partial y}\right)+KA_{44}\left(\frac{\partial^2 w_0}{\partial y^2}+\frac{\partial \phi_y}{\partial y}\right)+\\
&\mathcal{N}(w)+q-\left(\frac{\partial Q_x^P}{\partial x}+\frac{\partial Q_y^P}{\partial y}\right)=I_0\frac{\partial^2 w_0}{\partial t^2}
\end{aligned}
$$

(6.2-24c)

$$
\begin{aligned}
&B_{11}\left(\frac{\partial^2 u_0}{\partial x^2}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x^2}\right)+B_{12}\left(\frac{\partial^2 v_0}{\partial y\partial x}+\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial y\partial x}\right)+\\
&B_{16}\left(\frac{\partial^2 u_0}{\partial y\partial x}+\frac{\partial^2 v_0}{\partial x^2}+\frac{\partial^2 w_0}{\partial x^2}\frac{\partial w_0}{\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial y\partial x}\right)+\\
&D_{11}\frac{\partial^2 \phi_x}{\partial x^2}+D_{12}\frac{\partial^2 \phi_y}{\partial y\partial x}+D_{16}\left(\frac{\partial^2 \phi_x}{\partial x\partial y}+\frac{\partial^2 \phi_y}{\partial x^2}\right)+\\
&B_{16}\left(\frac{\partial^2 u_0}{\partial x\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x\partial y}\right)+B_{26}\left(\frac{\partial^2 v_0}{\partial y^2}+\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial y^2}\right)+\\
&B_{66}\left(\frac{\partial^2 u_0}{\partial y^2}+\frac{\partial^2 v_0}{\partial x\partial y}+\frac{\partial^2 w_0}{\partial x\partial y}\frac{\partial w_0}{\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial y^2}\right)+\\
&D_{16}\frac{\partial^2 \phi_x}{\partial x\partial y}+D_{26}\frac{\partial^2 \phi_y}{\partial y^2}+D_{66}\left(\frac{\partial^2 \phi_x}{\partial y^2}+\frac{\partial^2 \phi_y}{\partial y\partial x}\right)-\\
&KA_{55}\left(\frac{\partial w_0}{\partial x}+\phi_x\right)-KA_{45}\left(\frac{\partial w_0}{\partial y}+\phi_y\right)-\\
&\left(\frac{\partial M_{xx}^T}{\partial x}+\frac{\partial M_{xy}^T}{\partial y}\right)-\left(\frac{\partial M_{xx}^P}{\partial x}+\frac{\partial M_{xy}^P}{\partial y}-Q_x^P\right)\\
&=I_2\frac{\partial^2 \phi_x}{\partial t^2}+I_1\frac{\partial^2 u_0}{\partial t^2}
\end{aligned}
$$

(G. 2-24d)

$$
\begin{aligned}
&B_{16}\left(\frac{\partial^2 u_0}{\partial x^2}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x^2}\right)+B_{26}\left(\frac{\partial^2 v_0}{\partial y\partial x}+\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial y\partial x}\right)+\\
&B_{66}\left(\frac{\partial^2 u_0}{\partial y\partial x}+\frac{\partial^2 v_0}{\partial x^2}+\frac{\partial^2 w_0}{\partial x^2}\frac{\partial w_0}{\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial y\partial x}\right)+\\
&D_{16}\frac{\partial^2 \phi_x}{\partial x^2}+D_{26}\frac{\partial^2 \phi_y}{\partial y\partial x}+D_{66}\left(\frac{\partial^2 \phi_x}{\partial x\partial y}+\frac{\partial^2 \phi_y}{\partial x^2}\right)+
\end{aligned}
$$

$$
\begin{aligned}
&B_{12}\left(\frac{\partial^2 u_0}{\partial x\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x\partial y}\right)+B_{22}\left(\frac{\partial^2 v_0}{\partial y^2}+\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial y^2}\right)+\\
&B_{26}\left(\frac{\partial^2 u_0}{\partial y^2}+\frac{\partial^2 v_0}{\partial x\partial y}+\frac{\partial^2 w_0}{\partial x\partial y}\frac{\partial w_0}{\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial y^2}\right)+\\
&D_{12}\frac{\partial^2 \phi_x}{\partial x\partial y}+D_{22}\frac{\partial^2 \phi_y}{\partial y^2}+D_{26}\left(\frac{\partial^2 \phi_x}{\partial y^2}+\frac{\partial^2 \phi_y}{\partial x\partial y}\right)-\\
&KA_{45}\left(\frac{\partial w_0}{\partial x}+\phi_x\right)-KA_{44}\left(\frac{\partial w_0}{\partial y}+\phi_y\right)-\\
&\left(\frac{\partial M_{xy}^T}{\partial x}+\frac{\partial M_{yy}^T}{\partial y}\right)-\left(\frac{\partial M_{xy}^P}{\partial x}+\frac{\partial M_{yy}^P}{\partial y}-Q_y^P\right)\\
&=I_2\frac{\partial^2 \phi_y}{\partial t^2}+I_1\frac{\partial^2 v_0}{\partial t^2}
\end{aligned}
$$

(6.2-24e)

# Example: Cylindrical bending in FSDT $\left(\frac{\partial}{\partial y} = 0\right)$

$$A_{11}\frac{\partial^2 u_0}{\partial x^2} + A_{16}\frac{\partial^2 v_0}{\partial x^2} + B_{11}\frac{\partial^2 \phi_x}{\partial x^2} + B_{16}\frac{\partial^2 \phi_y}{\partial x^2} - \frac{\partial N_{xx}^T}{\partial x} - \frac{\partial N_{xx}^P}{\partial x} = I_0\frac{\partial^2 u_0}{\partial t^2} + I_1\frac{\partial^2 \phi_x}{\partial t^2} \tag{a}$$

$$A_{16}\frac{\partial^2 u_0}{\partial x^2} + A_{66}\frac{\partial^2 v_0}{\partial x^2} + B_{16}\frac{\partial^2 \phi_x}{\partial x^2} + B_{66}\frac{\partial^2 \phi_y}{\partial x^2} - \frac{\partial N_{xy}^T}{\partial x} - \frac{\partial N_{xy}^P}{\partial x} = I_0\frac{\partial^2 v_0}{\partial t^2} + I_1\frac{\partial^2 \phi_y}{\partial t^2} \tag{b}$$

$$B_{11}\frac{\partial^2 u_0}{\partial x^2} + B_{16}\frac{\partial^2 v_0}{\partial x^2} + D_{11}\frac{\partial^2 \phi_x}{\partial x^2} + D_{16}\frac{\partial^2 \phi_y}{\partial x^2} - KA_{55}\left(\frac{\partial w_0}{\partial x} + \phi_x\right) - KA_{45}\phi_y - \frac{\partial M_{xx}^T}{\partial x} - \frac{\partial M_{xx}^P}{\partial x} + Q_x^P = I_2\frac{\partial^2 \phi_x}{\partial t^2} + I_1\frac{\partial^2 u_0}{\partial t^2} \tag{c}$$

$$B_{16}\frac{\partial^2 u_0}{\partial x^2} + B_{66}\frac{\partial^2 v_0}{\partial x^2} + D_{16}\frac{\partial^2 \phi_x}{\partial x^2} + D_{66}\frac{\partial^2 \phi_y}{\partial x^2} - KA_{44}\phi_y - KA_{45}\left(\frac{\partial w_0}{\partial x} + \phi_x\right) - \frac{\partial M_{xy}^T}{\partial x} - \frac{\partial M_{xy}^P}{\partial x} + Q_y^P = I_2\frac{\partial^2 \phi_y}{\partial t^2} + I_1\frac{\partial^2 v_0}{\partial t^2} \tag{d}$$

$$KA_{55}\left(\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial \phi_x}{\partial x}\right) + KA_{45}\frac{\partial \phi_y}{\partial x} + \frac{\partial}{\partial x}\left(N_{xx}\frac{\partial w_0}{\partial x}\right) + q - \frac{\partial Q_x^P}{\partial x} = I_0\frac{\partial^2 w_0}{\partial t^2} \tag{e}$$

## 5.2.5 Laminate Stiffnesses for Selected Laminates

### I. Single-Layer Plates

#### I-a *Single Isotropic Layer*

$$A_{11} = \frac{Eh}{1-\nu^2}, \quad A_{12} = \nu A_{11}, \quad A_{22} = A_{11}, \quad A_{66} = \frac{1-\nu}{2} A_{11}, \quad A_{44} = A_{55} = \frac{1-\nu}{2} A_{11}$$

$$D_{11} = \frac{Eh^3}{12(1-\nu^2)}, \quad D_{12} = \nu D_{11}, \quad D_{22} = D_{11}, \quad D_{66} = \frac{1-\nu}{2} D_{11} \qquad (3.5.1)$$

(6.2-25)

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & \nu A_{11} & 0 \\ \nu A_{11} & A_{11} & 0 \\ 0 & 0 & \frac{1-\nu}{2} A_{11} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} \quad \text{(lb/in.)}$$

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & \nu D_{11} & 0 \\ \nu D_{11} & D_{11} & 0 \\ 0 & 0 & \frac{1-\nu}{2} D_{11} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} \quad \text{(lb-in/in.)}$$

(6.2-26)

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = K \frac{1-\nu}{2} \begin{bmatrix} A_{11} & 0 \\ 0 & A_{11} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \end{Bmatrix} \quad \text{(lb-in)}$$

$$N_{xx}^T = N_{yy}^T = \frac{E\alpha}{(1-\nu)} \int_{-\frac{h}{2}}^{\frac{h}{2}} \Delta T \, dz, \quad M_{xx}^T = M_{yy}^T = \frac{E\alpha}{(1-\nu)} \int_{-\frac{h}{2}}^{\frac{h}{2}} \Delta T z \, dz$$

(6.2-27)

## I-b *Single Specially Orthotropic Layer*

$$A_{11} = Q_{11}h, \quad A_{12} = Q_{12}h, \quad A_{22} = Q_{22}h$$
$$A_{66} = Q_{66}h, \quad A_{44} = Q_{44}h, \quad A_{55} = Q_{55}h \qquad (6.2\text{-}28)$$
$$D_{11} = \frac{Q_{11}h^3}{12}, \quad D_{12} = \frac{Q_{12}h^3}{12}, \quad D_{22} = \frac{Q_{22}h^3}{12}, \quad D_{66} = \frac{Q_{66}h^3}{12}$$

where

$$Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}}, \quad Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}, \quad Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}} \qquad (6.2\text{-}29)$$
$$Q_{66} = G_{12}, \quad Q_{44} = G_{23}, \quad Q_{55} = G_{13}$$

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = h \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix}$$

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \frac{h^3}{12} \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} \qquad (6.2\text{-}30)$$

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = Kh \begin{bmatrix} Q_{44} & 0 \\ 0 & Q_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \end{Bmatrix}$$

$$\begin{Bmatrix} N_{xx}^T \\ N_{yy}^T \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} \int_{-\frac{h}{2}}^{\frac{h}{2}} \Delta T \, dz$$

$$\begin{Bmatrix} M_{xx}^T \\ M_{yy}^T \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} \int_{-\frac{h}{2}}^{\frac{h}{2}} \Delta T z \, dz$$

(6.2-31)

I_C *Single Generally Orthotropic Layer*

$$A_{ij} = \bar{Q}_{ij} h, \quad D_{ij} = \frac{\bar{Q}_{ij} h^3}{12}, \quad A_{44} = h\bar{Q}_{44}, \; A_{55} = h\bar{Q}_{55}$$

(6.2-32)

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix}$$

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix}$$

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = K \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \end{Bmatrix}$$

(6.2-33)

$$\begin{Bmatrix} N_{xx}^T \\ N_{yy}^T \\ N_{xy}^T \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \alpha_{xx} \\ \alpha_{yy} \\ 2\alpha_{xy} \end{Bmatrix} \int_{-\frac{h}{2}}^{\frac{h}{2}} \Delta T dz$$

(6.2-34)

If the temperature increment is linear through the layer thickness, $\Delta T = T_0 + zT_1$,

$$\begin{Bmatrix} N_{xx}^T \\ N_{yy}^T \\ N_{xy}^T \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \alpha_{xx} \\ \alpha_{yy} \\ 2\alpha_{xy} \end{Bmatrix} T_0 h$$

$$\begin{Bmatrix} M_{xx}^T \\ M_{yy}^T \\ M_{xy}^T \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \alpha_{xx} \\ \alpha_{yy} \\ 2\alpha_{xy} \end{Bmatrix} \frac{T_1 h^3}{12}$$

(6.2-35)

## II - Symmetric Laminates

### II-a Symmetric Laminates with Multiple Isotropic Layers

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{11} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix}$$

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{11} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix}$$

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = K \begin{bmatrix} A_{44} & 0 \\ 0 & A_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \end{Bmatrix}$$

(6.2-36)

II-b Symmetric Laminates with Multiple Specially Orthotropic Layers

Eqs. are as the same as the (6.2-36)

II-c Symmetric Laminates with Multiple Generally Orthotropic Layers

A and D are completely full.

III- **Antisymmetric Laminates**

(in general)

(6.2-37)

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} \quad (3.$$

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} \quad (3.$$

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = K \begin{bmatrix} A_{44} & 0 \\ 0 & A_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \end{Bmatrix} \quad (3$$

Ⅲ_a **Antisymmetric Cross-ply Laminates**

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + \begin{bmatrix} B_{11} & 0 & 0 \\ 0 & -B_{11} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} \quad (3.$$

(6.2-38)

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & 0 & 0 \\ 0 & -B_{11} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} \quad (3.$$

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = K \begin{bmatrix} A_{44} & 0 \\ 0 & A_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \end{Bmatrix} \quad ($$

Ⅲ_b **Antisymmetric Angle-ply Laminates**

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + \begin{bmatrix} 0 & 0 & B_{16} \\ 0 & 0 & B_{26} \\ B_{16} & B_{26} & 0 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix}$$

(6.2-39)

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & B_{16} \\ 0 & 0 & B_{26} \\ B_{16} & B_{26} & 0 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix}$$

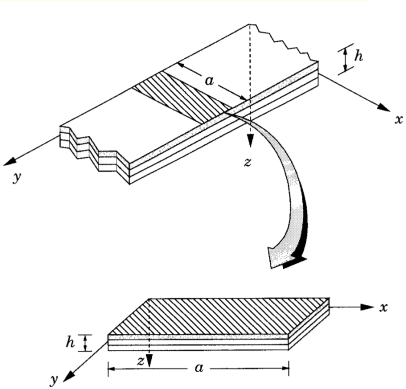
$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = K \begin{bmatrix} A_{44} & 0 \\ 0 & A_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \end{Bmatrix}$$

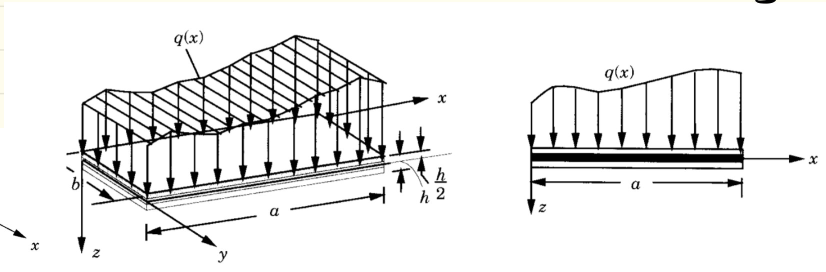
# Chapter VII One-Dimensional Analysis of Laminated Composite Plates

## 7.1- Introduction

**Beam**: The with of a plate is small compared to the length of the plate and $q(x)$. — Plane Stress

**Cylindrical bending**: Plate is very long along the $y$ and finite along the $x$ and $q(x)$. $\left(\frac{\partial}{\partial y}=0\right)$ — Plane strain




Beam

Cylindrical

governing equations →

- exact Solution — Satisfie gov. Eqs. & b.c & initial c. exactly
- numerical solution — Satisfie 〃 〃 〃 approximately

FEM, Finite Difference, Boundary Element

exact Solution

- closed-form (Analytical) — can be expressed in finite number of terms $\left( u(x) = 2 - x^2 + \sin \pi x \right)$
- infinite Series — $\left( u(x) = \sum_{i=1}^{\infty} a_i \sin(i\pi x) \right)$

$$u(x) = \sum_{i=1}^{N} a_i \sin(i\pi x)$$

exact-Analytical Solution (approximate)

# 7.2. Analysis of Laminated Beam Using CLPT

## 7.2-1 Governing Equation

Consider only the bending of the beam ($\{N\}=0$). Therefor

Constitutive Eq. is:

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = -[D] \begin{Bmatrix} \dfrac{\partial^2 w_0}{\partial x^2} \\ \dfrac{\partial^2 w_0}{\partial y^2} \\ 2\dfrac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix} \qquad (7.2\text{-}1)$$

Of course for bending of a beam we have:

$$M_y = M_{xy} = 0 \qquad 7.2\text{-}2)$$

$$\Rightarrow \begin{Bmatrix} \dfrac{\partial^2 w_0}{\partial x^2} \\ \dfrac{\partial^2 w_0}{\partial y^2} \\ 2\dfrac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix} = -\begin{bmatrix} D^*_{11} & D^*_{12} & D^*_{16} \\ D^*_{21} & D^*_{22} & D^*_{26} \\ D^*_{16} & D^*_{26} & D^*_{66} \end{bmatrix} \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} \qquad (7.2\text{-}3)$$

If the beam is long enough we can neglect the effect of Poisson's ration and shearing deformation. Thus

$$w_o = w_o(x, t) \qquad (7.2\text{-}3)$$

$$\boxed{\frac{\partial^2 w_o}{\partial x^2} = -D_{11}^* M_{xx}} \qquad (7.2\text{-}4)$$

Constitutive Eq. of CLPT Beam.