Composites — Lesson 18 — بسم الله الرحمن الرحيم

6.1-4- Laminate Constitutive Equations

(6.1-10)

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = [A] \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} + [B] \begin{Bmatrix} \varepsilon_x^{(1)} \\ \varepsilon_y^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} \quad (6.1-28)$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = [B] \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} + [D] \begin{Bmatrix} \varepsilon_x^{(1)} \\ \varepsilon_y^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} \quad (6.1-29)$$

$$\boxed{\begin{Bmatrix} \bar{N} \\ \bar{M} \end{Bmatrix} = \left[\begin{array}{c|c} A & B \\ \hline B & D \end{array}\right] \begin{Bmatrix} \varepsilon^o \\ \hline \varepsilon^{(1)} \end{Bmatrix}} \quad (6.1-30)$$

$$\begin{Bmatrix} \bar{N} \\ \bar{M} \end{Bmatrix} = \begin{Bmatrix} N \\ M \end{Bmatrix} + \begin{Bmatrix} N^T \\ M^T \end{Bmatrix} + \begin{Bmatrix} N^P \\ M^P \end{Bmatrix} \qquad (6.1\text{-}31)$$

$$\{N^T\} = \sum_{k=1}^{N} \int_{z_k}^{z_{k-1}} [\bar{Q}]_k \{\bar{\alpha}\}_k \Delta T \, dz \qquad (6.1\text{-}32)$$

$$\{M^T\} = \sum_{k=1}^{N} \int_{z_k}^{z_{k-1}} [\bar{Q}]_k \{\bar{\alpha}\}_k \Delta T \, z \, dz$$

$$\{N^P\} = \sum_{k=1}^{N} \int_{z_k}^{z_{k-1}} [\bar{e}]_k \{\epsilon\}_k \, dz \qquad (6.1\text{-}33)$$

$$\{M^P\} = \sum_{k=1}^{N} \int_{z_k}^{z_{k-1}} [\bar{e}]_k \{\epsilon\}_k \, dz$$

$\{N^P\}$ and $\{M^P\}$ are the piezoelectric resultants.

6.1.5. Equations of motions in Terms of Displacements

we had:

$$\{N\} = [A]\begin{Bmatrix} \dfrac{\partial u_0}{\partial x} + \frac{1}{2}\left(\dfrac{\partial w_0}{\partial x}\right)^2 \\ \dfrac{\partial v_0}{\partial y} + \frac{1}{2}\left(\dfrac{\partial w_0}{\partial y}\right)^2 \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} + \dfrac{\partial w_0}{\partial x}\dfrac{\partial w_0}{\partial y} \end{Bmatrix} - [B]\begin{Bmatrix} \dfrac{\partial^2 w_0}{\partial x^2} \\ \dfrac{\partial^2 w_0}{\partial y^2} \\ 2\dfrac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix} \quad (6.1\text{-}34)$$

$$\{M\} = [B]\{\ \text{"}\ \} - [D]\{\ \text{"}\ \} \quad (6.1\text{-}35)$$

substituting these equations in the governing equations (6.1-21) we have:

$$
\begin{aligned}
&A_{11}\left(\frac{\partial^2 u_0}{\partial x^2}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x^2}\right)+A_{12}\left(\frac{\partial^2 v_0}{\partial x\partial y}+\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial x\partial y}\right)\\
&+A_{16}\left(\frac{\partial^2 u_0}{\partial x\partial y}+\frac{\partial^2 v_0}{\partial x^2}+\frac{\partial^2 w_0}{\partial x^2}\frac{\partial w_0}{\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x\partial y}\right)\\
&-B_{11}\frac{\partial^3 w_0}{\partial x^3}-B_{12}\frac{\partial^3 w_0}{\partial x\partial y^2}-2B_{16}\frac{\partial^3 w_0}{\partial x^2\partial y}\\
&+A_{16}\left(\frac{\partial^2 u_0}{\partial x\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x\partial y}\right)+A_{26}\left(\frac{\partial^2 v_0}{\partial y^2}+\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial y^2}\right)\\
&+A_{66}\left(\frac{\partial^2 u_0}{\partial y^2}+\frac{\partial^2 v_0}{\partial x\partial y}+\frac{\partial^2 w_0}{\partial x\partial y}\frac{\partial w_0}{\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial y^2}\right)\\
&-B_{16}\frac{\partial^3 w_0}{\partial x^2\partial y}-B_{26}\frac{\partial^3 w_0}{\partial y^3}-2B_{66}\frac{\partial^3 w_0}{\partial x\partial y^2}\\
&-\left(\frac{\partial N_{xx}^T}{\partial x}+\frac{\partial N_{xy}^T}{\partial y}\right)=I_0\frac{\partial^2 u_0}{\partial t^2}-I_1\frac{\partial^3 w_0}{\partial x\partial t^2}
\end{aligned}
$$

(6.1-36a)

$$
\begin{aligned}
&A_{16}\left(\frac{\partial^2 u_0}{\partial x^2}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x^2}\right)+A_{26}\left(\frac{\partial^2 v_0}{\partial x\partial y}+\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial x\partial y}\right)\\
&+A_{66}\left(\frac{\partial^2 u_0}{\partial x\partial y}+\frac{\partial^2 v_0}{\partial x^2}+\frac{\partial^2 w_0}{\partial x^2}\frac{\partial w_0}{\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x\partial y}\right)\\
&-B_{16}\frac{\partial^3 w_0}{\partial x^3}-B_{26}\frac{\partial^3 w_0}{\partial x\partial y^2}-2B_{66}\frac{\partial^3 w_0}{\partial x^2\partial y}\\
&+A_{12}\left(\frac{\partial^2 u_0}{\partial x\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x\partial y}\right)+A_{22}\left(\frac{\partial^2 v_0}{\partial y^2}+\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial y^2}\right)\\
&+A_{26}\left(\frac{\partial^2 u_0}{\partial y^2}+\frac{\partial^2 v_0}{\partial x\partial y}+\frac{\partial^2 w_0}{\partial x\partial y}\frac{\partial w_0}{\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial y^2}\right)\\
&-B_{12}\frac{\partial^3 w_0}{\partial x^2\partial y}-B_{22}\frac{\partial^3 w_0}{\partial y^3}-2B_{26}\frac{\partial^3 w_0}{\partial x\partial y^2}\\
&-\left(\frac{\partial N_{xy}^T}{\partial x}+\frac{\partial N_{yy}^T}{\partial y}\right)=I_0\frac{\partial^2 v_0}{\partial t^2}-I_1\frac{\partial^3 w_0}{\partial y\partial t^2}
\end{aligned}
$$

(6.1-36b)

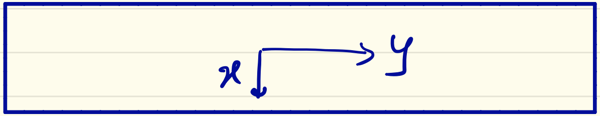
$$
\begin{aligned}
&B_{11}\left(\frac{\partial^3 u_0}{\partial x^3}+\frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 w_0}{\partial x^2}+\frac{\partial w_0}{\partial x}\frac{\partial^3 w_0}{\partial x^3}\right)+B_{12}\left(\frac{\partial^3 v_0}{\partial x^2\partial y}+\frac{\partial^2 w_0}{\partial x\partial y}\frac{\partial^2 w_0}{\partial x\partial y}\right.\\
&\left.+\frac{\partial w_0}{\partial y}\frac{\partial^3 w_0}{\partial x^2\partial y}\right)+B_{16}\left(\frac{\partial^3 u_0}{\partial x^2\partial y}+\frac{\partial^3 v_0}{\partial x^3}+\frac{\partial^3 w_0}{\partial x^3}\frac{\partial w_0}{\partial y}+2\frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 w_0}{\partial x\partial y}\right.\\
&\left.+\frac{\partial w_0}{\partial x}\frac{\partial^3 w_0}{\partial x^2\partial y}\right)-D_{11}\frac{\partial^4 w_0}{\partial x^4}-D_{12}\frac{\partial^4 w_0}{\partial x^2\partial y^2}-2D_{16}\frac{\partial^4 w_0}{\partial x^3\partial y}\\
&+2B_{16}\left(\frac{\partial^3 u_0}{\partial x^2\partial y}+\frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 w_0}{\partial x\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial^3 w_0}{\partial x^2\partial y}\right)+2B_{26}\left(\frac{\partial^3 v_0}{\partial x\partial y^2}+\frac{\partial^2 w_0}{\partial x\partial y}\frac{\partial^2 w_0}{\partial y^2}\right.\\
&\left.+\frac{\partial w_0}{\partial y}\frac{\partial^3 w_0}{\partial x\partial y^2}\right)+2B_{66}\left(\frac{\partial^3 u_0}{\partial x\partial y^2}+\frac{\partial^3 v_0}{\partial x^2\partial y}+\frac{\partial^3 w_0}{\partial x^2\partial y}\frac{\partial w_0}{\partial y}+\frac{\partial^2 w_0}{\partial x\partial y}\frac{\partial^2 w_0}{\partial x\partial y}\right.\\
&\left.+\frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 w_0}{\partial y^2}+\frac{\partial w_0}{\partial x}\frac{\partial^3 w_0}{\partial x\partial y^2}\right)-2D_{16}\frac{\partial^4 w_0}{\partial x^3\partial y}-2D_{26}\frac{\partial^4 w_0}{\partial x\partial y^3}\\
&-4D_{66}\frac{\partial^4 w_0}{\partial x^2\partial y^2}+B_{12}\left(\frac{\partial^3 u_0}{\partial x\partial y^2}+\frac{\partial^2 w_0}{\partial x\partial y}\frac{\partial^2 w_0}{\partial x\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial^3 w_0}{\partial x\partial y^2}\right)
\end{aligned}
$$

(6.1-36c)

$$
\begin{aligned}
&+B_{22}\left(\frac{\partial^3 v_0}{\partial y^3}+\frac{\partial^2 w_0}{\partial y^2}\frac{\partial^2 w_0}{\partial y^2}+\frac{\partial w_0}{\partial y}\frac{\partial^3 w_0}{\partial y^3}\right)+B_{26}\left(\frac{\partial^3 u_0}{\partial y^3}+\frac{\partial^3 v_0}{\partial x\partial y^2}\right.\\
&\left.+\frac{\partial^3 w_0}{\partial x\partial y^2}\frac{\partial w_0}{\partial y}+2\frac{\partial^2 w_0}{\partial x\partial y}\frac{\partial^2 w_0}{\partial y^2}+\frac{\partial w_0}{\partial x}\frac{\partial^3 w_0}{\partial y^3}\right)\\
&-D_{12}\frac{\partial^4 w_0}{\partial x^2\partial y^2}-D_{22}\frac{\partial^4 w_0}{\partial y^4}-2D_{26}\frac{\partial^4 w_0}{\partial x\partial y^3}+\mathcal{N}(w_0)+q\\
&-\left(\frac{\partial^2 M_{xx}^T}{\partial x^2}+2\frac{\partial^2 M_{xy}^T}{\partial y\partial x}+\frac{\partial^2 M_{yy}^T}{\partial y^2}\right)\\
&=I_0\frac{\partial^2 w_0}{\partial t^2}-I_2\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w_0}{\partial x^2}+\frac{\partial^2 w_0}{\partial y^2}\right)+I_1\frac{\partial^2}{\partial t^2}\left(\frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial y}\right)
\end{aligned}
$$

Example: (Cylindrical Bending)

In a plate when we have infinite demension along the y-axis and finite dimension along the x-axis and subjected to a transverse load $q(x)$



$$\left(\frac{\partial}{\partial y} = 0\right)$$

$$A_{11}\left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x^2}\right) + A_{16}\frac{\partial^2 v_0}{\partial x^2} - B_{11}\frac{\partial^3 w_0}{\partial x^3} - \frac{\partial N_{xx}^T}{\partial x} = I_0\frac{\partial^2 u_0}{\partial t^2} - I_1\frac{\partial^3 w_0}{\partial x \partial t^2}$$

$$A_{16}\left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x^2}\right) + A_{66}\frac{\partial^2 v_0}{\partial x^2} - B_{16}\frac{\partial^3 w_0}{\partial x^3} - \frac{\partial N_{xy}^T}{\partial x} = I_0\frac{\partial^2 v_0}{\partial t^2}$$

$$B_{11}\left(\frac{\partial^3 u_0}{\partial x^3} + \frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial w_0}{\partial x}\frac{\partial^3 w_0}{\partial x^3}\right) + B_{16}\frac{\partial^3 v_0}{\partial x^3} - D_{11}\frac{\partial^4 w_0}{\partial x^4} - \frac{\partial^2 M_{xx}^T}{\partial x^2}$$
$$+ \frac{\partial}{\partial x}\left(N_{xx}\frac{\partial w_0}{\partial x}\right) + q = I_0\frac{\partial^2 w_0}{\partial t^2} - I_2\frac{\partial^4 w_0}{\partial x^2 \partial t^2} + I_1\frac{\partial^3 u_0}{\partial x \partial t^2}$$

(6.1-37)

6-2- The First-Order Laminated Plate Theory (FSDT)

first-order Shear deformation laminated plate theory.

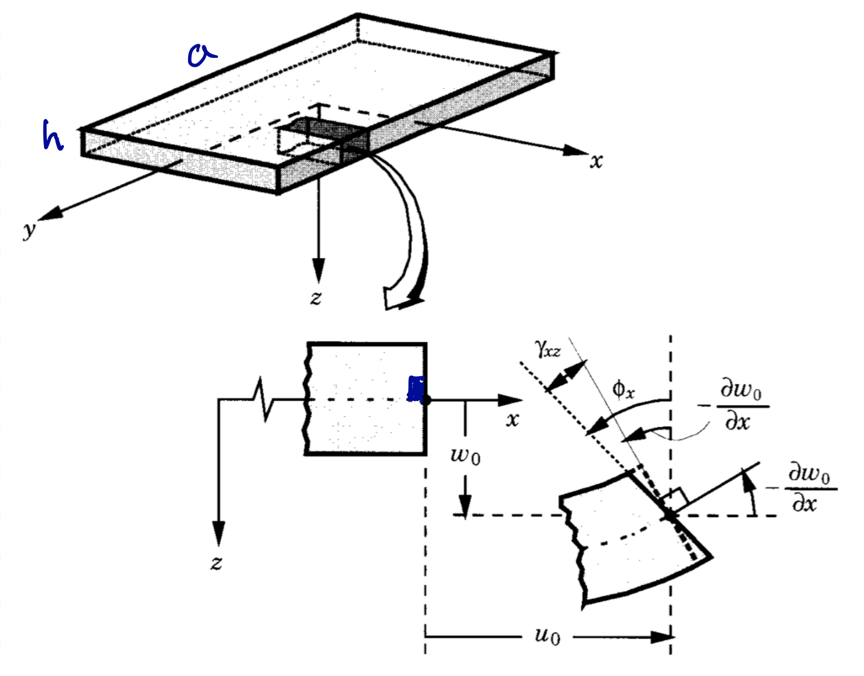
6.2-1- Displacements and Strains

$$u(x,y,z,t) = u_0(x,y,t) + z\,\phi_x(x,y,t)$$
$$v(x,y,z,t) = v_0(x,y,t) + z\,\phi_y(x,y,t) \qquad (6.2-1)$$
$$w(x,y,z,t) = w_0(x,y,t)$$

where $(u_0, v_0, w_0, \phi_x, \phi_y)$ are unknown functions to be determined. As before, (u_0, v_0, w_0) denote the displacements of a point on the plane $z=0$. Note that

$$\phi_x = \frac{\partial u}{\partial z} \quad , \quad \phi_y = \frac{\partial v}{\partial z} \qquad (6.2-2)$$

which indicate that ϕ_x and ϕ_y are the rotations of a transverse normal about the y- and x- axes.



if $\frac{a}{h} \geq 50$ then

$$\phi_x = -\frac{\partial w_0}{\partial x}, \quad \phi_y = -\frac{\partial w_0}{\partial y}$$

nonlinear Strains:

$$\varepsilon_x = \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 + z\frac{\partial \phi_x}{\partial x}$$ linear through the thickness

$$\varepsilon_y = \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2 + z\frac{\partial \phi_y}{\partial y}$$ 〃 〃 〃

$$\gamma_{xy} = \left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}\right) + z\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)$$ 〃 〃 〃

(6.2-4)

$$\left.\begin{aligned}\gamma_{xz} &= \frac{\partial w_0}{\partial x} + \phi_x \\ \gamma_{yz} &= \frac{\partial w_0}{\partial y} + \phi_y\end{aligned}\right\}$$ Constant through the thickness (approximation)

$$\varepsilon_z = 0$$

(6.2-5)

$$\begin{Bmatrix}\varepsilon_x \\ \varepsilon_y \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy}\end{Bmatrix} = \begin{Bmatrix}\varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{yz}^0 \\ \gamma_{xz}^0 \\ \gamma_{xy}^0\end{Bmatrix} + z\begin{Bmatrix}\varepsilon_x^{(1)} \\ \varepsilon_y^{(1)} \\ \gamma_{yz}^{(1)} \\ \gamma_{xz}^{(1)} \\ \gamma_{xy}^{(1)}\end{Bmatrix} = \begin{Bmatrix}\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 \\ \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2 \\ \frac{\partial w_0}{\partial x} + \phi_x \\ \frac{\partial w_0}{\partial y} + \phi_y \\ \frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} + \frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}\end{Bmatrix} + z\begin{Bmatrix}\frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ 0 \\ 0 \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\end{Bmatrix}$$

6.2.2 Equations of Motion

$$0 = \int_0^T (\delta U + \delta V - \delta K)\, dt \qquad (6.2\text{-}6)$$

$$\delta U = \int_{\Omega_0} \left\{ \int_{-h/2}^{-h/2} \left[\sigma_x (\delta \varepsilon_x^{(0)} + z\, \delta \varepsilon_x^{(1)}) + \sigma_y (\delta \varepsilon_y^{(0)} + z\, \delta \varepsilon_y^{(1)}) + \sigma_{xy} (\delta \gamma_{xy}^{(0)} + z\, \delta \gamma_{xy}^{(1)}) + \sigma_{xz}\, \delta \gamma_{xz}^{(0)} + \sigma_{yz}\, \delta \gamma_{yz}^{(0)} \right] dz \right\} dx\, dy$$

$$\delta V = - \int_{\Omega_0} \left[(q_b, q_t)\, \delta w_0 \right] dx\, dy - \int_{\Gamma_\sigma} \int_{-h/2}^{h/2} \left[\hat{\sigma}_n (\delta u_n + z\, \delta \phi_n) + \hat{\sigma}_{ns} (\delta u_s + z\, \delta \phi_s) + \hat{\sigma}_{nz}\, \delta w_0 \right] ds\, ds$$

$$\delta K = \int_{\Omega_0} \int_{-h/2}^{h/2} \rho_0 \left[(\dot{u}_0 + z \dot{\phi}_x)(\delta \dot{u}_0 + z\, \delta \dot{\phi}_x) + (\dot{v}_0 + z \dot{\phi}_y)(\delta \dot{v}_0 + z\, \delta \dot{\phi}_y) + \dot{w}_0\, \delta \dot{w}_0 \right] dz\, dx\, dy$$

$$(6.2\text{-}7)$$

Substituting (6.2-7) into (6.2-6) and integrating through the thickness we obtain:

$$0=\int_0^T\left\{\int_{\Omega_0}\left[N_{xx}\delta\varepsilon_{xx}^{(0)}+M_{xx}\delta\varepsilon_{xx}^{(1)}+N_{yy}\delta\varepsilon_{yy}^{(0)}+M_{yy}\delta\varepsilon_{yy}^{(1)}+N_{xy}\delta\gamma_{xy}^{(0)}+M_{xy}\delta\gamma_{xy}^{(1)}\right.\right.$$
$$+Q_x\delta\gamma_{xz}^{(0)}+Q_y\delta\gamma_{yz}^{(0)}-q\delta w_0-I_0\left(\dot u_0\delta\dot u_0+\dot v_0\delta\dot v_0+\dot w_0\delta\dot w_0\right)$$
$$\left.-I_1\left(\dot\phi_x\delta\dot u_0+\dot\phi_y\delta\dot v_0+\delta\dot\phi_x\dot u_0+\delta\dot\phi_y\dot v_0\right)-I_2\left(\dot\phi_x\delta\dot\phi_x+\dot\phi_y\delta\dot\phi_y\right)\right]dxdy$$
$$\left.-\int_{\Gamma_\sigma}\left(\hat N_{nn}\delta u_n+\hat N_{ns}\delta u_s+\hat M_{nn}\delta\phi_n+\hat M_{ns}\delta\phi_s+\hat Q_n\delta w_0\right)ds\right\}dt \qquad (3.4.9)$$

(6.2-8)

Shear Correction Factor

we had

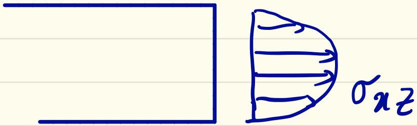
$$\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} dz \qquad (6.2-9)$$

In FSDT we assumed that shear stress is constant through the thickness

$$\sigma_{xz}=\frac{Q_x}{A} \qquad (6.2-10)$$

$A \rightarrow$ cross section area

But is well known from elementary theory of homogeneous beam that the transverse shear stress varies parabolically through the beam thickness.



This assumption make a mistake in calculating strain energy. It is often corrected by multiplying with a parameter K, called shear correction coefficient.

$$\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = K \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} dz \qquad (6.2\text{-}11)$$

$K = \dfrac{U^c}{U^F}$

$U^c \rightarrow$ strain energy based on actual shear stress distribution

$U^F \rightarrow$ " " " " FSDT

For example for rectangular cross section the actual shear stress distribution through the thickness is given by

$$\sigma_{xz}^{C} = \frac{3Q}{2bh}\left[1 - \left(\frac{2z}{h}\right)^2\right] \qquad (6.2\text{-}13)$$

$$\begin{cases} U^{C} = \dfrac{1}{2G_{13}} \displaystyle\int_A (\sigma_{xz}^{C})^2 dA = \dfrac{3Q^2}{5G_{13}bh} \\[2ex] U^{F} = \dfrac{1}{2G_{13}} \displaystyle\int_A (\sigma_{xz}^{F})^2 dA = \dfrac{Q^2}{3G_{13}bh} \end{cases} \qquad (6.2\text{-}14)$$

$$\Rightarrow K = \frac{U^C}{U^F} = \frac{5}{6} \qquad \text{(Rectangular)}$$