Composites Lesson 15 بسم الله الرحمن الرحيم

The variational operator ($\delta$) acts on the dependent variables (functionals) in the same way that the differential operator ($d$) acts on the independent variable.

$$\delta(F_1 F_2) = (\delta F_1) F_2 + F_1 (\delta F_2)$$

$$\delta(F_1^2) = 2F_1(\delta F_1)$$

$$\delta(y^2) = 2y\,\delta y$$

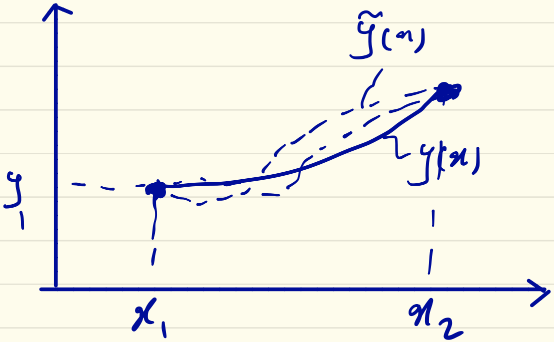
$$\frac{d}{dt}(\delta y) = \delta\left(\frac{dy}{dx}\right)$$ the property of Commutativity

Example:

$$y = x^2 + 3\sin(x)$$

$$\delta\left(x^2 + 3\sin(x)\right) = 0$$

$$\delta(y) \neq 0$$




$$\begin{cases} y(x_1) = y_1 \\ y(x_2) = y_2 \\ \delta y\big|_{x=x_1} = 0 \end{cases}$$

Example:

$$I(u) = \int_{x_1}^{x_2} \left( \frac{C_0}{2} u'^2 + \frac{C_1}{4} u'^4 + \frac{C_2}{2} u''^2 + f(x)\, u \right) dx$$

$\delta I = ?$

$$\delta I = \int \delta F\, dx = \int \left( C_0 u\,\delta u + C_1 u'^3 \delta u' + C_2 u'' \delta u'' + \underbrace{\cancel{\delta f}}_{0} u + f\,\delta u \right) dx$$

# 5.2. Boundary Conditions in plates

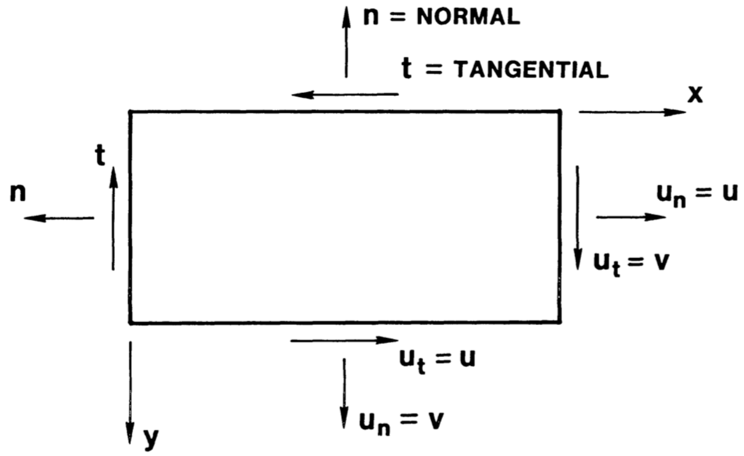
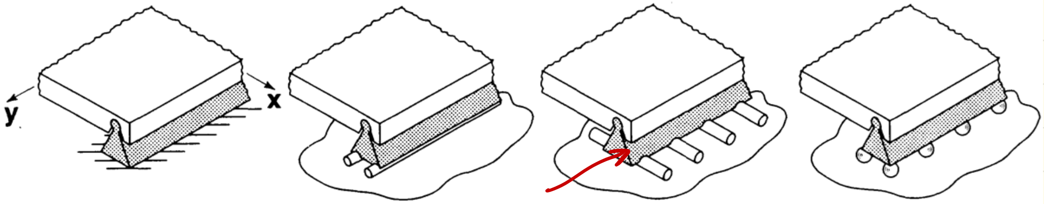


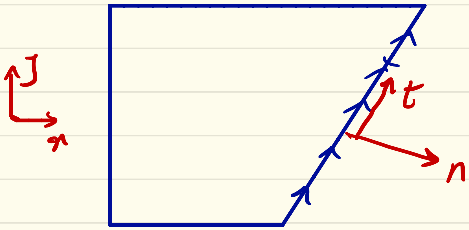

Figure 5-6 Boundary Condition Notation




| S1 | S2 | S3 | S4 |
|---|---|---|---|
| $w = 0$ | $w = 0$ | $w = 0$ | $w = 0$ |
| $M_x = 0$ | $M_x = 0$ | $M_x = 0$ | $M_x = 0$ |
| $u = 0$ | $N_x = 0$ | $u = 0$ | $N_x = 0$ |
| $v = 0$ | $v = 0$ | $N_{xy} = 0$ | $N_{xy} = 0$ |

(1) For the S1 condition, the triangular prism is fastened to a horizontal surface such that the prism cannot translate in either the x- or y-directions. Thus, $u = 0$ and $v = 0$. Hence, forces $N_x$ and $N_{xy}$ must exist.
(2) For the S2 condition, the triangular prism is supported on partially embedded y-direction roller bearings that permit translation in the x-direction, but none in the y-direction. Thus, $u \neq 0$ and $v = 0$. Hence, force $N_x$ must be zero, but $N_{xy}$ must exist.
(3) For the S3 condition, the triangular prism is supported on partially embedded x-direction roller bearings that permit translation in the y-direction, but none in the x-direction. Thus, $u = 0$ and $v \neq 0$. Hence, force $N_x$ must exist, but $N_{xy}$ must be zero.
(4) For the S4 condition, the triangular prism is supported on partially embedded spherical bearings that permit translation in any direction in the x-y plane. Thus, $u \neq 0$ and $v \neq 0$, so both forces $N_x$ and $N_{xy}$ must be zero.

| | | | | |
|---|---|---|---|---|
| S1: | $w=0$ | $M_n=0$ | $u_n=\bar{u}_n$ | $u_t=\bar{u}_t$ |
| S2: | $w=0$ | $M_n=0$ | $N_n=\bar{N}_n$ | $u_t=\bar{u}_t$ |
| S3: | $w=0$ | $M_n=0$ | $u_n=\bar{u}_n$ | $N_{nt}=\bar{N}_{nt}$ |
| S4: | $w=0$ | $M_n=0$ | $N_n=\bar{N}_n$ | $N_{nt}=\bar{N}_{nt}$ |
| C1: | $w=0$ | $w_{,n}=0$ | $u_n=\bar{u}_n$ | $u_t=\bar{u}_t$ |
| C2: | $w=0$ | $w_{,n}=0$ | $N_n=\bar{N}_n$ | $u_t=\bar{u}_t$ |
| C3: | $w=0$ | $w_{,n}=0$ | $u_n=\bar{u}_n$ | $N_{nt}=\bar{N}_{nt}$ |
| C4: | $w=0$ | $w_{,n}=0$ | $N_n=\bar{N}_n$ | $N_{nt}=\bar{N}_{nt}$ |



## 5.3. Orthoganal functions in Solving equations

$$\frac{dx}{dx} + 4\frac{d^2x}{dy^2} + 5x^2 + 6\,\mathrm{Sin}(x) = 0 \rightsquigarrow L(x) = 0$$

$$L(\ldots) = \frac{d}{dy}(\ldots) + 4\frac{d^2}{dy^2}(\ldots) + 5(\ldots)^2 + 6\,\mathrm{Sin}(\ldots)$$ operator

$$L(x,y) \equiv 5x^2 + 6y^2 + 3xy$$ two-variables operator

$u$ & $v$ are orthagonal based on $L(x,y)$ if: $L(u,v)=0$

dat operation: $\vec{a} \perp \vec{b}$ if $\vec{a} \cdot \vec{b} = 0$

$$\frac{1}{\pi}\int_0^{2\pi} \mathrm{Sin}(nx)\,\mathrm{Sin}(mx)\,dx = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$$

$$\int_0^a \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x\right) = \begin{cases} 0 & n \neq m \\ \frac{a}{2} & n = m \end{cases}$$

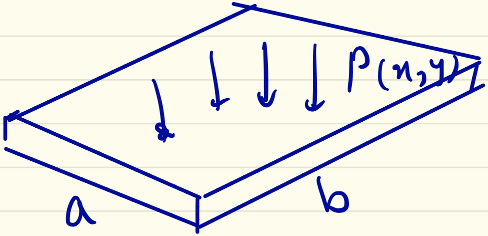
$$P(x) = a_1 \sin\left(\frac{\pi}{a}x\right) + a_2 \sin\left(\frac{2\pi}{a}x\right) + a_3 \sin\left(\frac{3\pi}{a}x\right) + \cdots -$$

$$\int_0^a P \sin\left(\frac{2\pi}{a}x\right)dx = 0 + a_2 \frac{a}{2} + 0 + \cdots \cdots$$

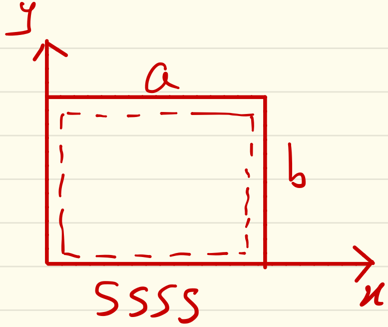
$$a_2 = \frac{2}{a}\int_0^a P \sin\left(\frac{2\pi}{a}x\right)dx$$

$$\boxed{a_i = \frac{2}{a}\int_0^a P(x) \sin\left(\frac{i\pi}{a}x\right) dx}$$

application example: (Navier approach)




$$\nabla^4 w = \frac{p}{D}$$




$$p = \sum_n \sum_m P_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

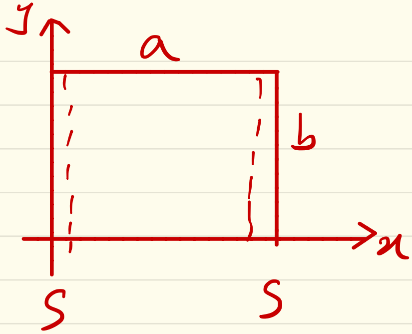
$$w = \sum_n \sum_m w_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$w_{mn}\left(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right)\left(\sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)\right)$$

$$= \frac{P_{mn}}{D} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$w_{mn}\left(\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right)^2 \cdot \pi^4 = P_{mn}/D$$

application example: (Levy approach)




$$\nabla^4 w = \frac{p}{D}$$

$$\begin{cases} w = \sum_m Y_m(y) \sin \frac{m\pi x}{a} \\ p = \sum_m P_m(y) \sin \frac{m\pi x}{a} \end{cases}$$

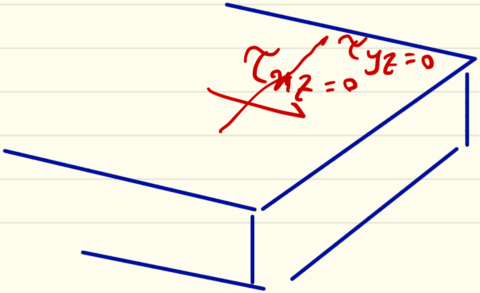
$$\sum_m \left( Y_m^{IV} - 2\frac{m^2\pi^2}{a^2} Y_m'' + \frac{m^4\pi^4}{a^4} Y_m \right) \sin\left(\frac{m\pi x}{a}\right) = \sum_m \frac{P_m(y)}{D} \sin \frac{m\pi x}{a}$$

$$\Rightarrow \quad Y_m^{IV}(y) - 2\frac{m^2\pi^2}{a^2} Y_m'' + \frac{m^4\pi^4}{a^4} Y_m = \frac{P_m(y)}{D}$$

# Chapter 6: Classical and First-Order Theories of Laminated Composite Plates

## General assumption

- The layers are perfectly bonded together (assumption).
- The material of each layer is linearly elastic and has three planes of material symmetry (i.e., orthotropic) (restriction).
- Each layer is of uniform thickness (restriction).
- The strains and displacements are small (restriction).
- The transverse shear stresses on the top and bottom surfaces of the laminate are zero (restriction).



1_ Classical laminated plate theory (CLPT)

which is an extension of the Kirchhoff (classical) plate theory

$$u(x,y,z,t) = u_0(x,y,t) - z\frac{\partial w_0}{\partial x}$$
$$v(x,y,z,t) = v_0(x,y,t) - z\frac{\partial w_0}{\partial y}$$
$$w(x,y,z,t) = w_0(x,y,t)$$

where $(u_0, v_0, w_0)$ are the displacement components along the $(x, y, z)$ Coordinate.

2_ The First-order Shear deformation Theory (FSDT)

$$u(x,y,z,t) = u_0(x,y,t) + z\phi_x(x,y,t)$$
$$v(x,y,z,t) = v_0(x,y,t) + z\phi_y(x,y,t)$$
$$w(x,y,z,t) = w_0(x,y,t)$$

## 3_ Third-order laminated plat Theory of Reddy (TLPT)

$$u(x,y,z,t) = u_0(x,y,t) + z\phi_x(x,y,t) + z^3\left(-\frac{4}{3h^2}\right)\left(\phi_x + \frac{\partial w_0}{\partial x}\right)$$

$$v(x,y,z,t) = v_0(x,y,t) + z\phi_y(x,y,t) + z^3\left(-\frac{4}{3h^2}\right)\left(\phi_y + \frac{\partial w_0}{\partial y}\right)$$

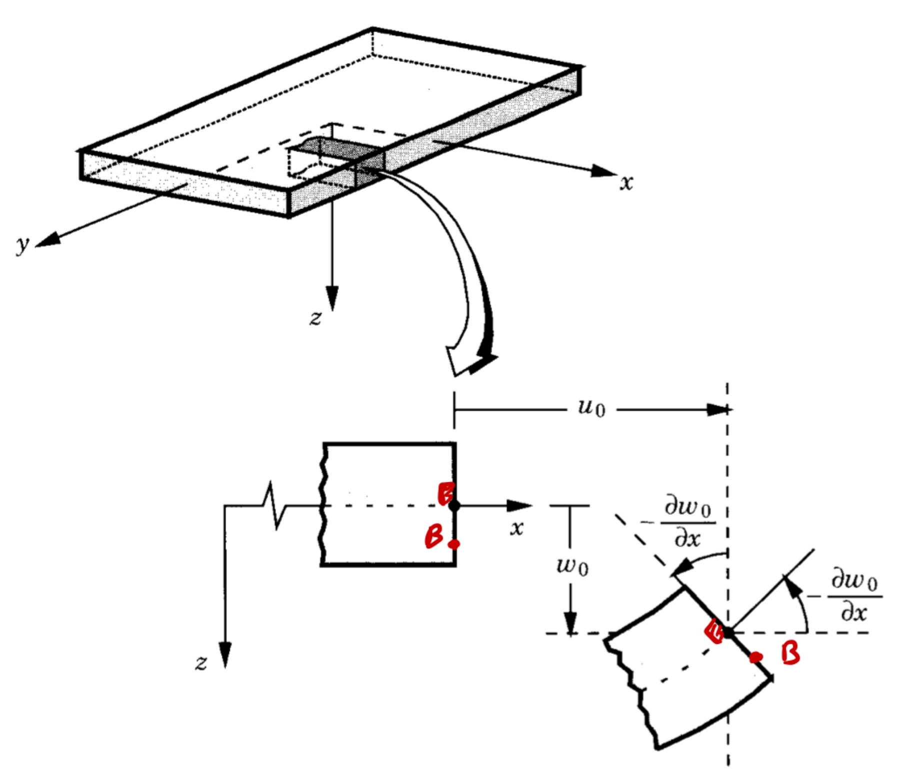
$$w(x,y,z,t) = w_0(x,y,t)$$

# 6.1_ The classical Laminated Plate Theory (CLPT)

(1) Straight lines perpendicular to the midsurface (i.e., transverse normals) before deformation remain straight after deformation.

(2) The transverse normals do not experience elongation (i.e., they are inextensible). $\varepsilon_z \approx 0$

(3) The transverse normals rotate such that they remain perpendicular to the midsurface after deformation.



x
y
z
$u_0$
B
B
x
$\frac{\partial w_0}{\partial x}$
$w_0$
z
B
B
$\frac{\partial w_0}{\partial x}$

## 6.1.1 Displacements and strains

$$u(x,y,z,t) = u_0(x,y,t) - z\frac{\partial w_0}{\partial x}$$

$$v(x,y,z,t) = v_0(x,y,t) - z\frac{\partial w_0}{\partial y}$$

$$w(x,y,z,t) = w_0(x,y,t)$$