Composites Lesson 17 بسم الله الرحمن الرحيم

By substituting (6.1-12,13,14) into (6.1-11) and integrating in z direction we have:

$$
\begin{aligned}
0 = \int_0^T \Bigg\{ \int_{\Omega_0} \Bigg[& N_{xx}\delta\varepsilon_{xx}^{(0)} + M_{xx}\delta\varepsilon_{xx}^{(1)} + N_{yy}\delta\varepsilon_{yy}^{(0)} + M_{yy}\delta\varepsilon_{yy}^{(1)} + N_{xy}\delta\gamma_{xy}^{(0)} \\
& + M_{xy}\delta\gamma_{xy}^{(1)} - q\delta w_0 - I_0\left(\dot{u}_0\delta\dot{u}_0 + \dot{v}_0\delta\dot{v}_0 + \dot{w}_0\delta\dot{w}_0\right) \\
& + I_1\left(\frac{\partial\delta\dot{w}_0}{\partial x}\dot{u}_0 + \frac{\partial\dot{w}_0}{\partial x}\delta\dot{u}_0 + \frac{\partial\delta\dot{w}_0}{\partial y}\dot{v}_0 + \frac{\partial\dot{w}_0}{\partial y}\delta\dot{v}_0\right) \\
& - I_2\left(\frac{\partial\dot{w}_0}{\partial x}\frac{\partial\delta\dot{w}_0}{\partial x} + \frac{\partial\dot{w}_0}{\partial y}\frac{\partial\delta\dot{w}_0}{\partial y}\right)\Bigg] dxdy \\
& \underbrace{- \int_{\Gamma_\sigma}\left(\hat{N}_{nn}\delta u_{0n} + \hat{N}_{ns}\delta u_{0s} - \hat{M}_{nn}\frac{\partial\delta w_0}{\partial n} - \hat{M}_{ns}\frac{\partial\delta w_0}{\partial s} + \hat{Q}_n\delta w_0\right) ds}_{\text{boundary expressions}} \Bigg\} dt
\end{aligned}
\qquad (6.1\text{-}15)
$$

where $q = q_b + q_t$ and

$$\left\{\begin{array}{c} N_{xx} \\ N_{yy} \\ N_{xy} \end{array}\right\} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left\{\begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{array}\right\} dz, \quad \left\{\begin{array}{c} M_{xx} \\ M_{yy} \\ M_{xy} \end{array}\right\} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left\{\begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{array}\right\} z\, dz$$

$$\left\{\begin{array}{c} \hat{N}_{nn} \\ \hat{N}_{ns} \end{array}\right\} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left\{\begin{array}{c} \hat{\sigma}_{nn} \\ \hat{\sigma}_{ns} \end{array}\right\} dz, \quad \left\{\begin{array}{c} \hat{M}_{nn} \\ \hat{M}_{ns} \end{array}\right\} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left\{\begin{array}{c} \hat{\sigma}_{nn} \\ \hat{\sigma}_{ns} \end{array}\right\} z\, dz$$

$$\left\{\begin{array}{c} I_0 \\ I_1 \\ I_2 \end{array}\right\} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left\{\begin{array}{c} 1 \\ z \\ z^2 \end{array}\right\} \rho_0\, dz, \qquad \hat{Q}_n = \int_{-\frac{h}{2}}^{\frac{h}{2}} \hat{\sigma}_{nz}\, dz$$

(6.1-16)

from (6.1-6) we can write:

$$\delta \varepsilon_x^0 = \frac{\partial \delta u_0}{\partial x} + \frac{\partial w_0}{\partial x}\frac{\partial \delta w_0}{\partial x}, \quad \delta \varepsilon_x^{(1)} = -\frac{\partial^2 \delta w_0}{\partial x^2}$$

$$\delta \varepsilon_y^0 = \frac{\partial \delta v_0}{\partial y} + \frac{\partial w_0}{\partial y}\frac{\partial \delta w_0}{\partial y}, \quad \delta \varepsilon_y^{(1)} = -\frac{\partial^2 \delta w_0}{\partial y^2}$$

$$\delta \gamma_{xy}^0 = \frac{\partial \delta u_0}{\partial y} + \frac{\partial \delta v_0}{\partial x} + \frac{\partial \delta w_0}{\partial x}\frac{\partial w_0}{\partial y} + \frac{\partial w_0}{\partial x}\frac{\partial \delta w_0}{\partial y}, \quad \delta \gamma_{xy}^{(1)} = -2\frac{\partial^2 \delta w_0}{\partial x \partial y} \qquad (6.1-17)$$

Bu substituting these, into (6.1-15) we have:

$$
\begin{aligned}
0 = \int_0^T \Bigg\{ \int_{\Omega_0} \Bigg[& - N_{xx,x}\delta u_0 - \left(N_{xx}\frac{\partial w_0}{\partial x}\right)_{,x} \delta w_0 - M_{xx,xx}\delta w_0 - N_{yy,y}\delta v_0 \\
& - \left(N_{yy}\frac{\partial w_0}{\partial y}\right)_{,y} \delta w_0 - M_{yy,yy}\delta w_0 - N_{xy,y}\delta u_0 - N_{xy,x}\delta v_0 \\
& - \left(N_{xy}\frac{\partial w_0}{\partial y}\right)_{,x} \delta w_0 - \left(N_{xy}\frac{\partial w_0}{\partial x}\right)_{,y} \delta w_0 - 2M_{xy,xy}\delta w_0 - q\delta w_0 \\
& + I_0 \left(\ddot{u}_0 \delta u_0 + \ddot{v}_0 \delta v_0 + \ddot{w}_0 \delta w_0 \right) - I_2 \left(\frac{\partial^2 \ddot{w}_0}{\partial x^2} + \frac{\partial^2 \ddot{w}_0}{\partial y^2} \right) \delta w_0 \\
& + I_1 \left(\frac{\partial \ddot{u}_0}{\partial x}\delta w_0 - \frac{\partial \ddot{w}_0}{\partial x}\delta u_0 + \frac{\partial \ddot{v}_0}{\partial y}\delta w_0 - \frac{\partial \ddot{w}_0}{\partial y}\delta v_0 \right) \Bigg] dxdy \\
+ \oint_\Gamma \Bigg[& N_{xx} n_x \delta u_0 + \left(N_{xx}\frac{\partial w_0}{\partial x}\right) n_x \delta w_0 - M_{xx} n_x \frac{\partial \delta w_0}{\partial x} + M_{xx,x} n_x \delta w_0 \\
& + N_{yy} n_y \delta v_0 + \left(N_{yy}\frac{\partial w_0}{\partial y}\right) n_y \delta w_0 - M_{yy} n_y \frac{\partial \delta w_0}{\partial y} + M_{yy,y} n_y \delta w_0 \\
& - M_{xy} n_x \frac{\partial \delta w_0}{\partial y} + M_{xy,x} n_y \delta w_0 - M_{xy} n_y \frac{\partial \delta w_0}{\partial x} + M_{xy,y} n_x \delta w_0 \\
& + N_{xy} n_y \delta u_0 + N_{xy} n_x \delta v_0 + N_{xy}\frac{\partial w_0}{\partial y} n_x \delta w_0 + N_{xy}\frac{\partial w_0}{\partial x} n_y \delta w_0 \Bigg] ds \\
- \int_{\Gamma_\sigma} & \left(\hat{N}_{nn}\delta u_{0n} + \hat{N}_{ns}\delta u_{0s} - \hat{M}_{nn}\frac{\partial \delta w_0}{\partial n} - \hat{M}_{ns}\frac{\partial \delta w_0}{\partial s} + \hat{Q}_n \delta w_0 \right) ds \\
+ \oint_\Gamma & \left[-I_1 \left(\ddot{u}_0 n_x + \ddot{v}_0 n_y \right) + I_2 \left(\frac{\partial \ddot{w}_0}{\partial x} n_x + \frac{\partial \ddot{w}_0}{\partial y} n_y \right) \right] \delta w_0 \; ds \Bigg\} dt \qquad (3.3.22)
\end{aligned}
$$

(6.1_18)

And we can rewrite it as:

$$
\begin{aligned}
0 = \int_0^T \Bigg\{ \int_{\Omega_0} \Bigg[& -\left(N_{xx,x} + N_{xy,y} - I_0\ddot{u}_0 + I_1\frac{\partial \ddot{w}_0}{\partial x}\right)\delta u_0 \\
& -\left(N_{xy,x} + N_{yy,y} - I_0\ddot{v}_0 + I_1\frac{\partial \ddot{w}_0}{\partial y}\right)\delta v_0 \\
& -\Bigg(M_{xx,xx} + 2M_{xy,xy} + M_{yy,yy} + \mathcal{N}(w_0) + q \\
& - I_0\ddot{w}_0 - I_1\frac{\partial \ddot{u}_0}{\partial x} - I_1\frac{\partial \ddot{v}_0}{\partial y} + I_2\frac{\partial^2 \ddot{w}_0}{\partial x^2} + I_2\frac{\partial^2 \ddot{w}_0}{\partial y^2}\Bigg)\delta w_0\Bigg] dxdy \\
+ \int_{\Gamma_\sigma} \Bigg[& \overbrace{(N_{xx}n_x + N_{xy}n_y)}^{\vec{N}\cdot\vec{n} = \hat{N}_n}\delta u_0 + \overbrace{(N_{xy}n_x + N_{yy}n_y)}^{N_{ns}}\delta v_0 \\
& + \Bigg(M_{xx,x}n_x + M_{xy,y}n_x + M_{yy,y}n_y + M_{xy,x}n_y + \mathcal{P}(w_0) \\
& - I_1\ddot{u}_0 n_x - I_1\ddot{v}_0 n_y + I_2\frac{\partial \ddot{w}_0}{\partial x}n_x + I_2\frac{\partial \ddot{w}_0}{\partial y}n_y\Bigg)\delta w_0 \\
& - (M_{xx}n_x + M_{xy}n_y)\frac{\partial \delta w_0}{\partial x} - (M_{xy}n_x + M_{yy}n_y)\frac{\partial \delta w_0}{\partial y}\Bigg] ds \\
- \int_{\Gamma_\sigma} & \left(\hat{N}_{nn}\delta u_{0n} + \hat{N}_{ns}\delta u_{0s} - \hat{M}_{nn}\frac{\partial \delta w_0}{\partial n} - \hat{M}_{ns}\frac{\partial \delta w_0}{\partial s} + \hat{Q}_n\delta w_0\right) ds \Bigg\} dt
\end{aligned}
$$

(6.1-19)

For linear analysis, we set $\mathcal{N}$ and $\mathcal{P}$ are zero.

where

$$
\mathcal{N}(w_0) = \frac{\partial}{\partial x}\left(N_{xx}\frac{\partial w_0}{\partial x} + N_{xy}\frac{\partial w_0}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w_0}{\partial x} + N_{yy}\frac{\partial w_0}{\partial y}\right)
$$

$$
\mathcal{P}(w_0) = \left(N_{xx}\frac{\partial w_0}{\partial x} + N_{xy}\frac{\partial w_0}{\partial y}\right)n_x + \left(N_{xy}\frac{\partial w_0}{\partial x} + N_{yy}\frac{\partial w_0}{\partial y}\right)n_y
$$

(6.1-20)

The Euler-Lagrange equations are obtained by setting the coefficients of δu_0, δv_0, δw_0 in Ω_0 to zero seperately:

$$\delta u_0: \quad \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0 \frac{\partial^2 u_0}{\partial t^2} - I_1 \frac{\partial^2}{\partial t^2}\left(\frac{\partial w_0}{\partial x}\right) \qquad (6.1\text{-}21)$$

$$\delta v_0: \quad \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = I_0 \frac{\partial^2 v_0}{\partial t^2} - I_1 \frac{\partial^2}{\partial t^2}\left(\frac{\partial w_0}{\partial y}\right)$$

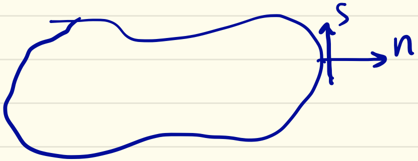
$$\delta w_0: \quad \frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + \mathcal{N}(w_0) + q = I_0 \frac{\partial^2 w_0}{\partial t^2} - I_2 \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial y^2}\right) + I_1 \frac{\partial^2}{\partial t^2}\left(\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y}\right)$$

Equations of Motion for CLPT Plate

In boundary expressions (6.1-19) the parameters in virtual variation form give us primary variables and coefficients of those variations indicate secondary variables.

$u_n, u_s, w_0, \frac{\partial w_0}{\partial n}, \frac{\partial w_0}{\partial s}$ Primary Variables (essential b.c.) geometrical b.c.

$N_n, N_{ns}, Q_n, M_n, M_{ns}$ Secondary Variables (natural b.c.) force b.c.



(6.1-22)

natural b.c.:

$$N_n - \hat{N}_n = 0 \quad , N_{ns} - \hat{N}_{ns} = 0 , \quad Q_n - \hat{Q}_n = 0$$

$$M_n - \hat{M}_n = 0 \quad , M_{ns} - \hat{M}_{ns} = 0 \qquad (6.1\text{-}23)$$

which by considering the coeffision of δw_0

we can say:

$$Q_n = \left(M_{x,x} + M_{xy,y} - I_1 \ddot{u}_0 + I_2 \frac{\partial \ddot{w}_0}{\partial x}\right) n_x + \qquad (6.1\text{-}24)$$

$$\left(M_{y,y} + M_{xy,x} - I_1 \ddot{v}_0 + I_2 \frac{\partial \ddot{w}_0}{\partial y}\right) n_y + P(w_0)$$

We note that the Eqs. (6.1-21) have the total spatial differential order of eight. In other words, if the equations are expressed in terms of the displacements (u_0, v_0, w_0), they would contain second-order spatial derivatives of u_0 and v_0 and fourth-order spatial derivatives of w_0. Hence, the classical laminated plate theory is said to be an eighth-order theory. So, we only have eight integral constants. This implies that there should be only eight boundary conditions. But, Eq (6.1-22) shows five essential and five natural b.c. giving a total of ten boundary conditions. To eliminate this discrepancy, in Eq. (6.1-19) the following integral can be changed by using integrating by part technique.

$$-\oint_{\Gamma} M_{ns}\frac{\partial \delta w_o}{\partial s}\, ds = \oint_{\Gamma} \frac{\partial M_{ns}}{\partial s}\, \delta w_o\, ds - \left[M_{ns}\, \delta w_o\right]_{\Gamma} \tag{6.1-25}$$

The term in the square bracket is zero since the end points of a closed curve coincide. This term now must be added to Q_n (because it is a coefficient of δw_o):

$$V_n = Q_n + \frac{\partial M_{ns}}{\partial s} = \hat{Q}_n \tag{6.1-26}$$

which should be balaned by the applied force $\hat{Q}_n$.
This boundary condition, $V_n = \hat{Q}_n$, is known as the Kirchhoff free-edge condition.

$u_n, u_s, w_0, \frac{\partial w_0}{\partial n}$ Primary var. (essential b.c.) (6.1-27)

N_n, N_{ns}, V_n, M_n secondary var. (natural b.c.)