Composites Lesson 16 بسم الله الرحمن الرحيم

The nonlinear strains are given by:

$$E_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right]$$

$$E_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] \qquad (6.1\text{-}2)$$

$$E_{zz} = \frac{\partial w}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right]$$

$$E_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)$$

$$E_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial z}\right)$$

$$E_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z}\right)$$

If the components of the displacement gradients are the order ϵ:

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z} = O(\epsilon) \qquad (6.1\text{-}3)$$

then the small strain assumption implies that terms of the order ϵ^2 are negligible in the strains. Terms of order ϵ^2 are

$$\left(\frac{\partial u}{\partial x}\right)^2, \left(\frac{\partial u}{\partial y}\right)^2, \left(\frac{\partial u}{\partial z}\right)^2, \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right), \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial z}\right), \left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial z}\right) \qquad (6.1\text{-}4)$$

$$\left(\frac{\partial v}{\partial x}\right)^2, \left(\frac{\partial v}{\partial y}\right)^2, \left(\frac{\partial v}{\partial z}\right)^2, \left(\frac{\partial v}{\partial x}\right)\left(\frac{\partial v}{\partial y}\right), \left(\frac{\partial v}{\partial x}\right)\left(\frac{\partial v}{\partial z}\right), \left(\frac{\partial v}{\partial y}\right)\left(\frac{\partial v}{\partial z}\right)$$

$$\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial z}\right), \left(\frac{\partial w}{\partial y}\right)\left(\frac{\partial w}{\partial z}\right), \left(\frac{\partial w}{\partial z}\right)^2$$

If the rotations $\frac{\partial w_0}{\partial x}$ and $\frac{\partial w_0}{\partial y}$ of transverse normals are moderate (say 10°-15°), then the following terms are small but not negligible compared to ϵ:

$$\left(\frac{\partial w}{\partial x}\right)^2, \left(\frac{\partial w}{\partial y}\right)^2, \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}$$

Thus for small strains and moderate rotations cases the strain-displacement relations take the form:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \qquad (6.1\text{-}5)$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2$$

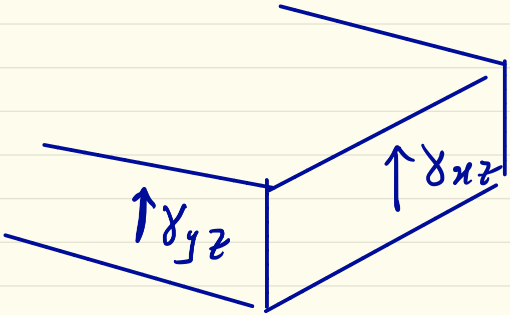
$$\varepsilon_{zz} = \frac{\partial w}{\partial z}$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)$$

$$\varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \quad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

Substituting (6.1-1) into (6.1-5) we have:

$$
\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 - z\frac{\partial^2 w_0}{\partial x^2} \\
\varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}\right) - z\frac{\partial^2 w_0}{\partial x \partial y} \\
\varepsilon_{yy} &= \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2 - z\frac{\partial^2 w_0}{\partial y^2} \\
\varepsilon_{xz} &= \frac{1}{2}\left(-\frac{\partial w_0}{\partial x} + \frac{\partial w_0}{\partial x}\right) = 0 \\
\varepsilon_{yz} &= \frac{1}{2}\left(-\frac{\partial w_0}{\partial y} + \frac{\partial w_0}{\partial y}\right) = 0 \\
\varepsilon_{zz} &= 0
\end{aligned}
\qquad (6.1\text{-}6)
$$



The strains in Eqs. (6.1-6) are called the Von Karman strains and the associated plate theory is termed the Von Karman plat theory.

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z \begin{Bmatrix} \varepsilon_x^{(1)} \\ \varepsilon_y^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} \qquad (6.1\text{-}7)$$

$$\{\varepsilon\}^0 = \begin{Bmatrix} \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 \\ \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2 \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y} \end{Bmatrix} \qquad (6.1\text{-}8)$$

membrane strain

$$\{\varepsilon\}^{(1)} = \begin{Bmatrix} -\frac{\partial^2 w_0}{\partial x^2} \\ -\frac{\partial^2 w_0}{\partial y^2} \\ -2\frac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix}$$

bending strain,
curvature

6.1-2 Lamina Constitutive Relations

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix}_k = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix}_k \begin{Bmatrix} \varepsilon_1 - \alpha_1 \Delta T \\ \varepsilon_2 - \alpha_2 \Delta T \\ \varepsilon_6 \end{Bmatrix}_k - \begin{bmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{32} \\ 0 & 0 & 0 \end{bmatrix}_k \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{Bmatrix}_k \quad (6.1\text{-}9)$$

$[e]_k$ is piezoelectric moduli, and $\{\epsilon\}$ is electric field.

In (x,y) Coordinates:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}^{(k)} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^{(k)} \left(\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} - \begin{Bmatrix} \alpha_{xx} \\ \alpha_{yy} \\ 2\alpha_{xy} \end{Bmatrix} \Delta T \right) - \begin{bmatrix} 0 & 0 & \bar{e}_{31} \\ 0 & 0 & \bar{e}_{32} \\ 0 & 0 & \bar{e}_{36} \end{bmatrix}^{(k)} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \end{Bmatrix}^{(k)} \quad (6.1\text{-}10)$$

6.1-3- Equations of Motion

In dynamic problems:

$$I = \int L\,dt = \int (\Pi - K)\,dt = \text{extremum}$$

$$\Rightarrow \delta I = 0 \longrightarrow \int (\delta\Pi - \delta K)\,dt = 0$$

$$\int_0^T (\delta U + \delta V - \delta K)\,dt = 0 \qquad (6.1\text{-}11)$$

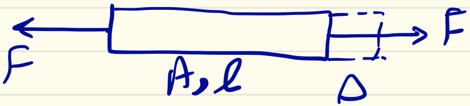
δU : virtual strain energy (Volume integral of δU_0)

δV : virtual work done by applied forces

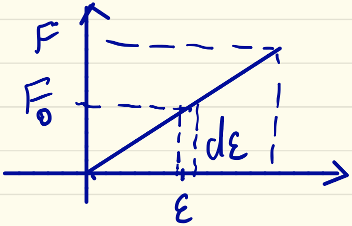
δK : virtual kinetic energy

Relation (6.1-11) is the dynamic version of the principle of virtual work.

δU virtual strain energy



$$W = F.\Delta \longrightarrow W_0 = \frac{W}{V} = \frac{F}{A}\cdot\frac{\Delta}{l}$$



$$W_0 = \sigma.\varepsilon \longrightarrow dW_0 = \sigma.d\varepsilon$$

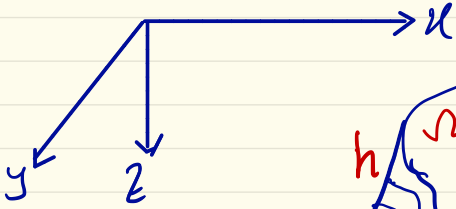
$$\delta U_0 = \sigma.\delta\varepsilon$$

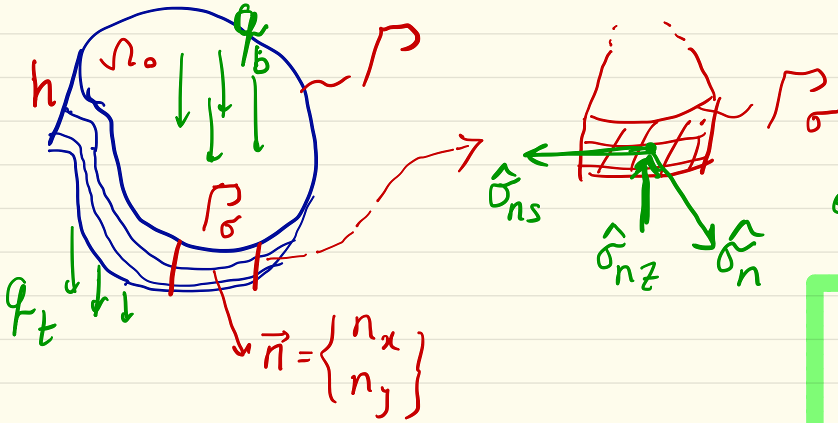
$$\delta U = \int \delta U_0 \, dV$$

$$\delta U = \int_V \{\sigma\}^T.\{\delta\varepsilon\}\, dV$$

$$= \int_{\Omega_0}\int_{-h/2}^{+h/2} (\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + 2\sigma_{xy}\delta\varepsilon_{xy})\, dz\, dx\, dy$$

$$\delta U = \int_{\Omega_0} \left\{ \int_{-h/2}^{+h/2} \left[\sigma_x \left(\delta\varepsilon_x^{\circ} + z\,\delta\varepsilon_x^{(1)} \right) + \sigma_y \left(\delta\varepsilon_y^{\circ} + z\,\delta\varepsilon_y^{(1)} \right) + \sigma_{xy} \left(\delta\gamma_{xy}^{\circ} + z\,\delta\gamma_{xy}^{(1)} \right) \right] dz \right\} dx\,dy \qquad (6.1\text{-}12)$$





external forces
loads

δV virtual work done by applied forces

$$V = \vec{F}\cdot\vec{u} \rightsquigarrow \delta V = \vec{F}\cdot\delta\vec{u}$$

$$\delta V = -\int_{\Omega_0}\left[q_b(x,y)\,\delta w\left(x,y,\frac{h}{2}\right) + q_t(x,y)\,\delta w\left(x,y,-\frac{h}{2}\right)\right]dx\,dy$$

$$-\int_{\Gamma_\sigma}\int_{-h/2}^{h/2}\left[\hat{\sigma}_n\,\delta u_n + \hat{\sigma}_{ns}\,\delta u_s + \hat{\sigma}_{nz}\,\delta w\right]dz\,ds$$

$$= \int_{\Omega_0}\left[(q_b + q_t)\,\delta w_0(x,y)\right]dx\,dy$$

$$-\int_{\Gamma_\sigma}\int_{-h/2}^{h/2}\left\{\hat{\sigma}_n\left(\delta u_{0n} - z\frac{\partial\delta w_0}{\partial n}\right) + \hat{\sigma}_{nn}\left(\delta u_{0s} - z\frac{\partial\delta w_0}{\partial s}\right)\right.$$

$$\left. + \hat{\sigma}_{nz}\,\delta w_0\right]dz\,ds \qquad (6.1\text{-}13)$$

where q_b is the distributed force at the bottom $(z=\frac{h}{2})$ of the laminate, q_t is the distributed force at the top $(z=-h/2)$ of the laminate, $(\hat{\sigma}_n, \hat{\sigma}_{ns}, \hat{\sigma}_{nz})$ are the specified stress components on the portion Γ_σ of the boundary Γ.

δK virtual kinetic energy

$$K = \int_V \frac{1}{2}\rho(\dot{u}^2 + \dot{v}^2 + \dot{w}^2)\, dV \longrightarrow \delta K = \int_V \rho(\dot{u}\,\delta\dot{u} + \dot{v}\,\delta\dot{v} + \dot{w}\,\delta\dot{w})\, dV$$

$$\delta K = \int_{\Omega_0}\int_{-h/2}^{h/2} \rho_0 \left[\left(\dot{u}_0 - z\frac{\partial \dot{w}_0}{\partial x}\right)\left(\delta\dot{u}_0 - z\frac{\partial \delta\dot{w}_0}{\partial x}\right) + \left(\dot{v}_0 - z\frac{\partial \dot{w}_0}{\partial y}\right)\left(\delta\dot{v}_0 - z\frac{\partial \delta\dot{w}_0}{\partial y}\right) + \dot{w}_0\,\delta\dot{w}_0\right] dz\,dx\,dy$$

(6.1-14)