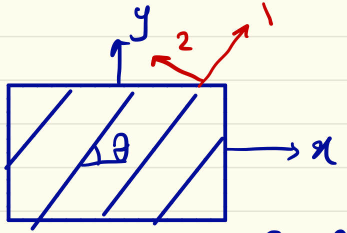
Composites — Lesson 14 — بسم الله الرحمن الرحيم

4.6- Engineering Constants for a Laminate

a- Single-layered



$$E_x = \frac{E_1}{m^4 + \left(\frac{E_1}{G_{12}} - 12\nu_{12}\right) n^2 m^2 + \frac{E_1}{E_2} n^4}$$

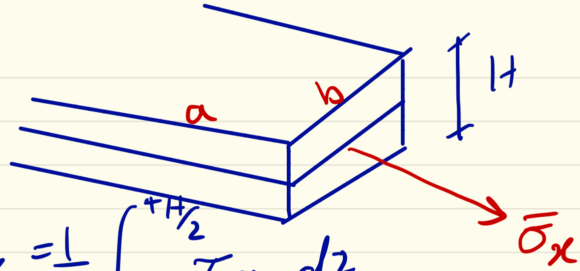
$m, n = \cos\theta, \sin\theta$

$\{\varepsilon_x\} = [\bar{S}]\{\sigma\}$

$\eta_{xy,y} = \frac{\gamma_{xy}}{\varepsilon_y}$ only $\sigma_y \neq 0$ and others are zero $= \frac{\bar{S}_{26}}{\bar{S}_{22}}$

$\eta_{x,xy} = \frac{\varepsilon_x}{\gamma_{xy}}$ only $\tau_{xy} \neq 0$ and others are zero $= \frac{\bar{S}_{16}}{\bar{S}_{66}}$

b- Multi-layered Laminates



$$\bar{\sigma}_x = \frac{1}{H}\int_{-H/2}^{H/2} \sigma_x \, dz \,, \quad \bar{\sigma}_y = \frac{1}{H}\int_{-H/2}^{+H/2} \sigma_y \, dz \,, \quad \bar{\tau}_{xy} = \frac{1}{H}\int_{-H/2}^{+H/2} \tau_{xy} \, dz$$

$$\bar{\sigma}_x = \frac{1}{H} N_x \,, \quad \bar{\sigma}_y = \frac{1}{H} N_y \,, \quad \bar{\tau}_{xy} = \frac{1}{H} N_{xy}$$

$$\{N\} = [A]\{\varepsilon\}^0 + [B]\{K\} \xrightarrow{\text{if } [B]=0} \{N\} = [A]\{\varepsilon\}^0 \Rightarrow \{\varepsilon\}^0 = [A]^{-1}\{N\}$$

$$\{\varepsilon\}^0 = [A]^{-1} H \{\bar{\sigma}\}$$

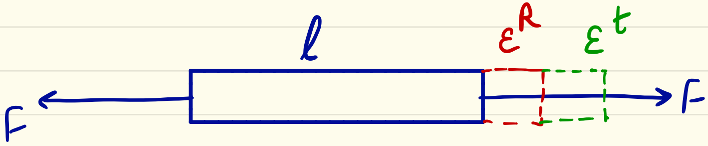
Effective compliance matrix for a Laminate

$$\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} = \overbrace{\begin{bmatrix} a_{11}H & a_{12}H & 0 \\ a_{12}H & a_{22}H & 0 \\ 0 & 0 & a_{66}H \end{bmatrix}} \begin{Bmatrix} \bar{\sigma}_x \\ \bar{\sigma}_y \\ \bar{\tau}_{xy} \end{Bmatrix}$$

$$\bar{E}_x = \frac{1}{a_{11} H}, \quad \bar{E}_y = \frac{1}{a_{22} H}, \quad \bar{G}_{xy} = \frac{1}{a_{66} H}$$

$$\bar{\nu}_{xy} = -\frac{a_{12}}{a_{11}}, \quad \bar{\nu}_{yx} = -\frac{a_{12}}{a_{22}}$$

4.7- Termal and Mechanical Stress Analysis



$$\varepsilon = \varepsilon^R + \varepsilon^t$$

$$\Delta^t = \alpha l \Delta t \quad , \quad \varepsilon^t = \alpha \Delta t$$

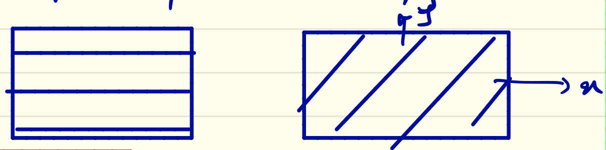
α: coefficient of Thermal Expansion (CTE)

$$\sigma_x = E_x \overbrace{(\varepsilon - \varepsilon^t)}^{\text{redundant strain}}$$

$$\{\sigma\} = [C]\left(\{\varepsilon\} - \{\varepsilon\}^t\right) \quad , \quad \{\varepsilon\}^t = \{\alpha\} \Delta t$$

For orthotropic materials in principle orthotropy direction we have:

$$\{\alpha\} = \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{Bmatrix}$$



$$\boxed{\begin{Bmatrix} \alpha_x \\ \alpha_y \\ \frac{1}{2}\alpha_{xy} \end{Bmatrix} = [T(\theta)]^{-1} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{Bmatrix}}$$

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{Bmatrix} = [\bar{Q}]\begin{Bmatrix}\varepsilon_1-\alpha_1\Delta t\\ \varepsilon_2-\alpha_2\Delta t\\ \gamma_{12}\end{Bmatrix}$$

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}_K = [\bar{Q}]_K\begin{Bmatrix}\varepsilon_x-\alpha_x\Delta t\\ \varepsilon_y-\alpha_y\Delta t\\ \gamma_{xy}-\alpha_{xy}\Delta t\end{Bmatrix}_K$$

$$\{N\} = [A]\{\varepsilon^o\} + [B]\{K\} - \begin{Bmatrix}N_x^t\\ N_y^t\\ N_{xy}^t\end{Bmatrix}$$

$$\begin{Bmatrix}N_x^t\\ N_y^t\\ N_{xy}^t\end{Bmatrix} = \sum\int_{z_{K-1}}^{z_K}[\bar{Q}]_K\begin{Bmatrix}\alpha_x\\ \alpha_y\\ \alpha_{xy}\end{Bmatrix}_K\Delta t\,dz = \sum_{K=1}^{N}[\bar{Q}]_K\{\alpha\}_K\Delta t\,(z_K-z_{K-1})$$

$$\{M\} = [B]\{\varepsilon\}^o + [D]\{K\} - \{M\}^t$$

$$\begin{Bmatrix}M_x^t\\ M_y^t\\ M_{xy}^t\end{Bmatrix} = \sum\int[\bar{Q}]_K\{\alpha\}_K z\Delta t\,dz = \sum_{K=1}^{N}[\bar{Q}]_K\{\alpha_K\}\Delta t\,\tfrac{1}{2}\left(z_K^2-z_{K-1}^2\right)$$

$$\left\{\begin{array}{c} \bar{N} \\ \hdashline \bar{M} \end{array}\right\} = \left[\begin{array}{c:c} A & B \\ \hdashline B & D \end{array}\right] \left\{\begin{array}{c} \varepsilon^{\circ} \\ \hdashline K \end{array}\right\}$$

$$\{\bar{N}\} = \{N\} + \{N\}^{t}$$

$$\{\bar{M}\} = \{M\} + \{M\}^{t}$$

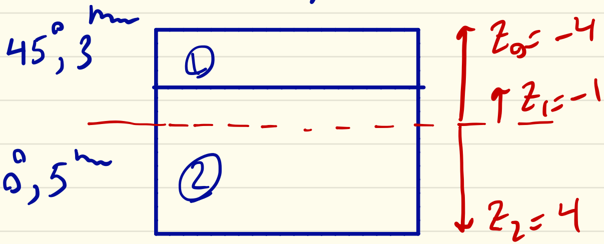
$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k = [\bar{Q}]_k \left(\{\varepsilon\}^{\circ} + z\{K\} - \{\alpha\}\Delta t \right)$$

Example 6: Find the residual stress in the example 1. The producing tempreture of the laminate is $125\,^\circ C$ and its application tempreture is $25\,^\circ C$.

$\alpha_1 = 7 \times 10^{-6}\ 1/^\circ C$

$\alpha_2 = 23 \times 10^{-6}\ 1/^\circ C$

$\Delta t = 25 - 125 = -100$



$$\begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix}_{0^\circ} = \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 7 \\ 23 \\ 0 \end{Bmatrix} \times 10^{-6} \quad 1/^\circ C$$

$$\begin{Bmatrix} \alpha_x \\ \alpha_y \\ \frac{1}{2}\alpha_{xy} \end{Bmatrix}_{45^\circ} = [T(45^\circ)]^{-1} \begin{Bmatrix} 7 \\ 23 \\ 0 \end{Bmatrix} \times 10^{-6}$$

$$\begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix}_{45^\circ} = \begin{Bmatrix} 15 \\ 15 \\ -16 \end{Bmatrix} 10^{-6} \quad 1/^\circ C$$

$$[\bar{Q}]_0 \{\alpha\}_0 \Delta t = \begin{Bmatrix} -15.6 \\ -5.09 \\ 0 \end{Bmatrix} 10^{-3}$$

$$[\bar{Q}]_{45^\circ} \{\alpha\}_{45^\circ} \Delta t = \begin{Bmatrix} -10.35 \\ -10.35 \\ -5.26 \end{Bmatrix} 10^{-3}$$

$$\{N\}^t = \sum \overbrace{[\bar{Q}]_k \{\alpha\}_k \Delta t}\,(z_k - z_{k-1}) = \begin{Bmatrix} -15.61 \\ -5.09 \\ 0 \end{Bmatrix} \times 10^{-3} \Big\{ [4-(-1)] + \begin{Bmatrix} -10.35 \\ -10.35 \\ -5.26 \end{Bmatrix} 10^{-3} [(-1)-(-4)]$$

$$= \begin{Bmatrix} -109.1 \\ -56.5 \\ -15.78 \end{Bmatrix} \times 10^{-3} \quad \frac{GN\text{-}mm}{m^2}$$

$$\{M\}^t = \sum \overbrace{[\bar{Q}]_k \{\alpha\}_k \Delta t}\,\tfrac{1}{2}(z_k^2 - z_{k-1}^2) = \begin{Bmatrix} -15.61 \\ -5.09 \\ 0 \end{Bmatrix} 10^{-3} \times \tfrac{1}{2}\left[(4^2-(-1)^2\right]$$

$$+ \begin{Bmatrix} -10.35 \\ -10.35 \\ -5.62 \end{Bmatrix} 10^{-3} \times \tfrac{1}{2}\left\{(-1)^2 - (-4)^2\right\} = \begin{Bmatrix} -39.45 \\ 39.45 \\ 39.45 \end{Bmatrix} 10^{-3} \quad \frac{GN\text{-}mm^2}{m^2}$$

$$\{\bar{N}\} = \{N\} + \{N\}^t = \{0\} + \{N\}^t$$

$$\{\bar{M}\} = \{M\} + \{M\}^t = \{0\} + \{M\}^t$$

$$\left\{\begin{array}{c}\bar{N}\\ \hline \bar{M}\end{array}\right\} = \left[\begin{array}{c|c} A & B \\ \hline B & D \end{array}\right] \left\{\begin{array}{c}\varepsilon^0\\ \hline K\end{array}\right\} \Rightarrow \left\{\begin{array}{c}\varepsilon^0\\ \hline K\end{array}\right\} = \left[\begin{array}{c|c} A & B \\ \hline B & D \end{array}\right]^{-1} \left\{\begin{array}{c}\bar{N}\\ \hline \bar{M}\end{array}\right\}$$

$$\begin{Bmatrix}\varepsilon_x^0\\ \varepsilon_y^0\\ \gamma_{xy}^0\end{Bmatrix} = \begin{Bmatrix}-8.14\\ -20.2\\ 6.99\end{Bmatrix}\times 10^{-4} \quad , \quad \begin{Bmatrix}K_x\\ K_y\\ K_{xy}\end{Bmatrix} = \begin{Bmatrix}0.58\\ -1\\ -2.55\end{Bmatrix}\times 10^{-4}$$

For finding stresses in each layer we have:

$$\{\varepsilon\}^R = \{\varepsilon\} - \{\varepsilon\}^t = \begin{Bmatrix}\varepsilon_x^0 + zK_x\\ \varepsilon_y^0 + zK_y\\ \gamma_{xy}^0 + zK_{xy}\end{Bmatrix} - \begin{Bmatrix}\alpha_x \Delta t\\ \alpha_y \Delta t\\ \alpha_{xy} \Delta t\end{Bmatrix}$$

$$\{\sigma\}_k = [\bar{Q}]_S \{\varepsilon\}^R$$

$$\begin{Bmatrix} \varepsilon_x^R \\ \varepsilon_y^R \\ \gamma_{xy}^R \end{Bmatrix}_{0^\circ} = \begin{Bmatrix} -13+0.58z \\ 2.8-1.0z \\ 6.99-2.35z \end{Bmatrix} \times 10^{-4}, \quad \begin{Bmatrix} \varepsilon_x^R \\ \varepsilon_y^R \\ \gamma_{xy}^R \end{Bmatrix}_{45^\circ} = \begin{Bmatrix} 6.86+0.58z \\ -5.2-1.0z \\ -9.01-2.35z \end{Bmatrix} \times 10^{-4}$$

For 0°- layer:

$z=4$ $$\{\sigma\}_0 = \begin{bmatrix} 20 & 0.7 & 0 \\ & 0.2 & 0 \\ \text{sym} & & 0.7 \end{bmatrix} \begin{Bmatrix} 1.18 \\ -1.2 \\ -2.4 \end{Bmatrix} \times 10^{-4} \ GN/m^2$$

$z=-1$ $$\{\sigma\}_0 = [\quad " \quad] \begin{Bmatrix} -1.72 \\ 3.8 \\ 9.34 \end{Bmatrix} \times 10^{-4} \ GN/m^2$$

For 45°- layer:

$z=-1$ $$\{\sigma\}_{45} = \begin{Bmatrix} -1.05 \\ -2.51 \\ -2.49 \end{Bmatrix} \times 10^{-3} \ GN/m^2$$

$z=-4$ $$\{\sigma\}_{45^\circ} = \begin{Bmatrix} -4.27 \\ 0.71 \\ -0.75 \end{Bmatrix} \times 10^{-3} \ GN/m^2$$