بسم الله الرحمن الرحيم — مواد مرکب — جلسه ۲۲

Calculation of stresses

$$\{\sigma\}_k = [\bar{Q}]_k \{\varepsilon\} \quad \{\varepsilon\} = \{\varepsilon\}^0 + z\{\varepsilon\}^{(1)}$$

$$\{\sigma\}_k = z[\bar{Q}]_k \begin{Bmatrix} -\dfrac{\partial^2 w_0}{\partial x^2} \\ -\dfrac{\partial^2 w_0}{\partial y^2} \\ -2\dfrac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix} \qquad (6.2\text{-}14)$$

با استفاده از $(6.2\text{-}3)$

$$\{\sigma\}_k = \frac{z}{b}[\bar{Q}]_k [D^*] \begin{Bmatrix} M \\ 0 \\ 0 \end{Bmatrix} \qquad (6.2\text{-}15)$$

$$\left(M_x = \frac{M}{b}\right)$$

$$\sigma_{xx}^{(k)}(x,z) = \frac{M(x)z}{b}\left(\bar{Q}_{11}^{(k)}D_{11}^* + \bar{Q}_{12}^{(k)}D_{12}^* + \bar{Q}_{16}^{(k)}D_{16}^*\right)$$

$$\sigma_{yy}^{(k)}(x,z) = \frac{M(x)z}{b}\left(\bar{Q}_{12}^{(k)}D_{11}^* + \bar{Q}_{22}^{(k)}D_{12}^* + \bar{Q}_{26}^{(k)}D_{16}^*\right)$$

$$\sigma_{xy}^{(k)}(x,z) = \frac{M(x)z}{b}\left(\bar{Q}_{16}^{(k)}D_{11}^* + \bar{Q}_{26}^{(k)}D_{12}^* + \bar{Q}_{66}^{(k)}D_{16}^*\right) \qquad (6.2\text{-}16)$$

ما در ابتدای کار فرض کردیم $\sigma_{xz} = \sigma_{zz} = 0$ و با این فرض معادلات فضای حالت را بدست آوردیم. اما در اینجا مقادیری برای این تنش ها از روابط زیر در CLPT قابل محاسبه است.

معادلات اساسی حاکم بر جسم

$$
\begin{aligned}
0 &= \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} \\
0 &= \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} \\
0 &= \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}
\end{aligned}
\qquad (6.2\text{-}17)
$$

پس داریم:

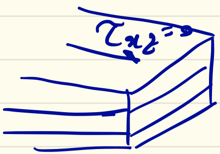
$$
\begin{aligned}
\sigma_{xz}^{(k)} &= -\int_{z_k}^{z} \left(\frac{\partial \sigma_{xx}^{(k)}}{\partial x} + \frac{\partial \sigma_{xy}^{(k)}}{\partial y} \right) dz + G^{(k)} \\
\sigma_{yz}^{(k)} &= -\int_{z_k}^{z} \left(\frac{\partial \sigma_{xy}^{(k)}}{\partial x} + \frac{\partial \sigma_{yy}^{(k)}}{\partial y} \right) dz + F^{(k)} \\
\sigma_{zz}^{(k)} &= -\int_{z_k}^{z} \left(\frac{\partial \sigma_{xz}^{(k)}}{\partial x} + \frac{\partial \sigma_{yz}^{(k)}}{\partial y} \right) dz + H^{(k)}
\end{aligned}
\qquad (6.2\text{-}18)
$$

دقت شود که این روابط برای هر لایه نوشته شده است. و مقادیر G^k و F^k، H^k ثابت های انتگرال برای هر لایه هستند.

چون مقادیر ما مستقل از y هستند پس رابطه τ_{yz} حذف می‌شود. و $\frac{\partial}{\partial y}=0$. پس داریم:

$$6.2\text{-}16 \Rightarrow \quad \sigma_{xz}^{(k)}(x,z) = -Q_x(x)\left(\bar{Q}_{11}^{(k)}D_{11}^{*}+\bar{Q}_{12}^{(k)}D_{12}^{*}+\bar{Q}_{16}^{(k)}D_{16}^{*}\right)\left(\frac{z^2-z_k^2}{2}\right)+G^{(k)}$$

$$\sigma_{zz}^{(k)}(x,z) = -\frac{dQ_x}{dx}\left(\bar{Q}_{11}^{(k)}D_{11}^{*}+\bar{Q}_{12}^{(k)}D_{12}^{*}+\bar{Q}_{16}^{(k)}D_{16}^{*}\right)\left(\frac{z^3-z_k^3}{6}\right)+H^{(k)} \qquad (6.2\text{-}19)$$

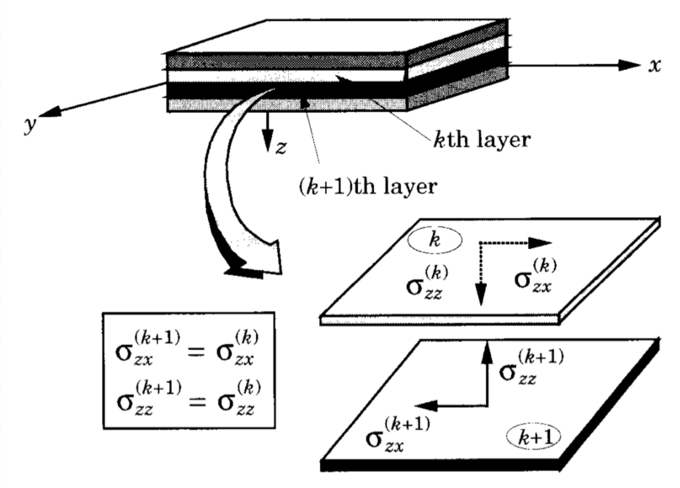


اگر سطح بالائی بدون تنش باشد (اگر تنشی وجود داشته باشد $H^{(1)}$ و $G^{(1)}$ قابل محاسبه است)

$$G^{(1)}=0 \ , \quad H^{(1)}=0$$

اما در سطوح بعدی باید

تعادلی بین تنش‌ها برقرار باشد



$$\sigma_{xz}^{(k)}(x, z_{k+1}) = \sigma_{xz}^{(k+1)}(x, z_{k+1}), \quad \sigma_{zz}^{(k)}(x, z_{k+1}) = \sigma_{zz}^{(k+1)}(x, z_{k+1})$$

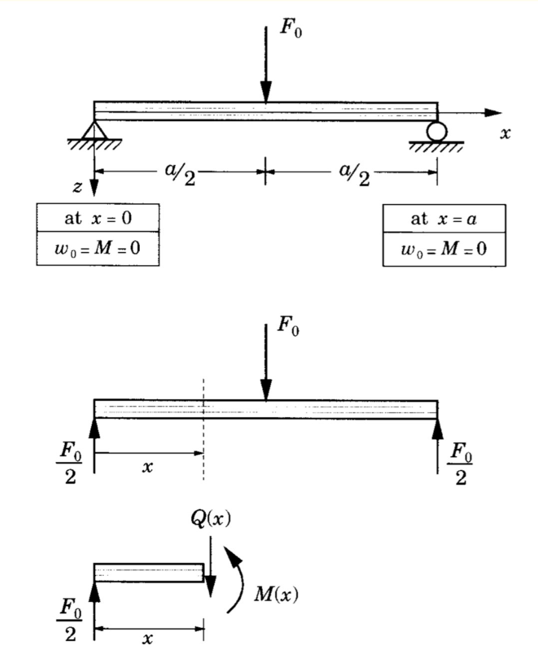
لذا می‌توان ثابت‌های انتگرال $k=2,3,\dots$ راحتی حساب نمود

$$G^{(k+1)} = -Q_x(x)\left(\bar{Q}_{11}^{(k)}D_{11}^* + \bar{Q}_{12}^{(k)}D_{12}^* + \bar{Q}_{16}^{(k)}D_{16}^*\right)\left(\frac{z_{k+1}^2 - z_k^2}{2}\right) + G^{(k)}$$
$$= \sigma_{xz}^{(k)}(x, z_{k+1}) \quad (4.2$$

(6.2-20)

$$H^{(k+1)} = -\frac{dQ_x}{dx}\left(\bar{Q}_{11}^{(k)}D_{11}^* + \bar{Q}_{12}^{(k)}D_{12}^* + \bar{Q}_{16}^{(k)}D_{16}^*\right)\left(\frac{z_{k+1}^3 - z_k^3}{6}\right) + H^{(k)}$$
$$= \sigma_{zz}^{(k)}(x, z_{k+1}) \quad (4.2$$

three-point-bending :مثال



$$M(x) = \frac{(F_0 b)x}{2}\ , \quad \text{for} \quad 0 \leq x \leq \frac{a}{2}$$

into Eq. (6.2.11a) and evaluating the inte

$$E^b_{xx} I_{yy} w_0(x) = -\frac{F_0 b x^3}{12} + c_1 x + c_2$$

evaluated using the boundary conditions

$$w_0(0) = 0, \quad \frac{dw_0}{dx}(a/2) = 0$$

$_2 = 0)$

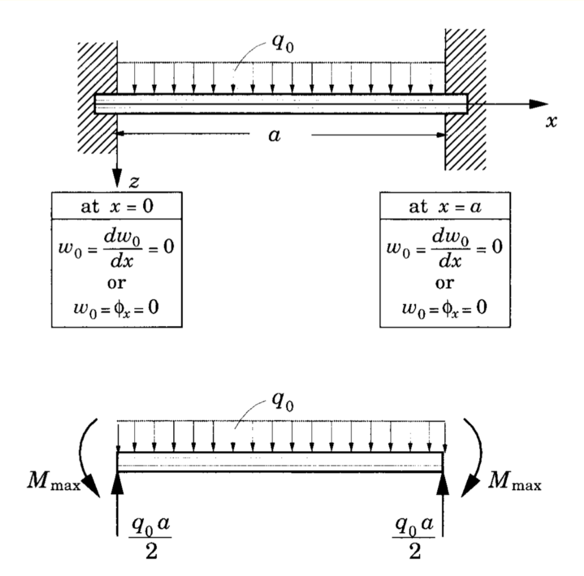
$$w_0(x) = \frac{F_0 b a^3}{48 E^b_{xx} I_{yy}} \left[3\left(\frac{x}{a}\right) - 4\left(\frac{x}{a}\right)^3 \right]$$

The maximum in-plane stress σ_{xx} occurs at $x = a/2$ $(M(a/2) = F_0 b a/4)$

$$\sigma^{(k)}_{xx}(a/2, z) = \frac{F_0 a z}{4} \left(\bar{Q}^{(k)}_{11} D^*_{11} + \bar{Q}^{(k)}_{12} D^*_{12} + \bar{Q}^{(k)}_{16} D^*_{16} \right)$$

مثال:

با استفاده از (6.2-11b) داریم:



$$E_{xx}^b I_{yy} w_0(x) = \frac{q_0 b x^4}{24} + c_1 \frac{x^3}{6} + c_2 \frac{x^2}{2} + c_3 x + c_4$$

beam. For the full beam case we have

$$w_0(0) = 0, \quad w_0(a) = 0, \quad \frac{dw_0}{dx}(0) = 0, \quad \frac{dw_0}{dx}(a) = 0$$

for the half beam model we have

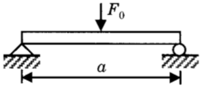
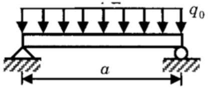
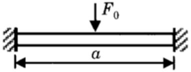
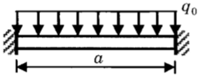
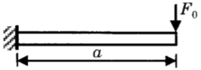
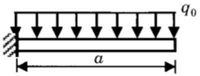
$$w_0(0) = 0, \quad \frac{dw_0}{dx}(0) = 0, \quad \frac{dw_0}{dx}\left(\frac{a}{2}\right) = 0, \quad Q\left(\frac{a}{2}\right) = \frac{dM}{dx} = -E_{xx} I_{yy} \frac{d^3 w_0}{dx^3}\left(\frac{a}{2}\right) = 0$$

در هر صورت

$$w_0(x) = \frac{q_0 b a^4}{24 E_{xx}^b I_{yy}} \left[\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)\right]^2$$

$$M(x) = -\frac{q_0 b a^2}{12}\left[1 - 6\left(\frac{x}{a}\right) + 6\left(\frac{x}{a}\right)^2\right], \quad M_{max} = -\frac{q_0 b a^2}{12}$$

$$\sigma_{xx}^{(k)}(0,z) = -\frac{q_0 a^2 z}{12}\left(Q_{11}^{(k)} D_{11}^{*} + Q_{12}^{(k)} D_{12}^{*} + Q_{16}^{(k)} D_{16}^{*}\right)$$

Laminated Beam	Deflection, $w_0(x)$	w_{max} and M_{max}
• *Hinged-Hinged*		
Central point load 	$\frac{c_1}{48}\left[3\left(\frac{x}{a}\right) - 4\left(\frac{x}{a}\right)^3\right]$	$w^c_{max} = \frac{1}{48}c_1$ $M^c_{max} = -\frac{1}{4}c_3$
Uniform load 	$\frac{c_2}{24}\left[\left(\frac{x}{a}\right) - 2\left(\frac{x}{a}\right)^3 + \left(\frac{x}{a}\right)^4\right]$	$w^c_{max} = \frac{5}{384}c_2$ $M^c_{max} = -\frac{1}{8}c_4$
• *Fixed-Fixed*		
Central point load 	$\frac{c_1}{48}\left[3\left(\frac{x}{a}\right)^2 - 4\left(\frac{x}{a}\right)^3\right]$	$w^c_{max} = \frac{1}{192}c_1$ $M^0_{max} = \frac{1}{8}c_3$
Uniform load 	$\frac{c_2}{24}\left[\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)\right]^2$	$w^c_{max} = \frac{1}{384}c_2$ $M^0_{max} = \frac{1}{12}c_4$
• *Fixed-Free*		
Point load at free end 	$\frac{c_1}{6}\left[3\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)^3\right]$	$w^a_{max} = \frac{1}{3}c_1$ $M^0_{max} = c_3$
Uniform load 	$\frac{c_2}{24}\left[6\left(\frac{x}{a}\right)^2 - 4\left(\frac{x}{a}\right)^3 + \left(\frac{x}{a}\right)^4\right]$	$w^a_{max} = \frac{1}{8}c_2$ $M^0_{max} = \frac{1}{2}c_4$

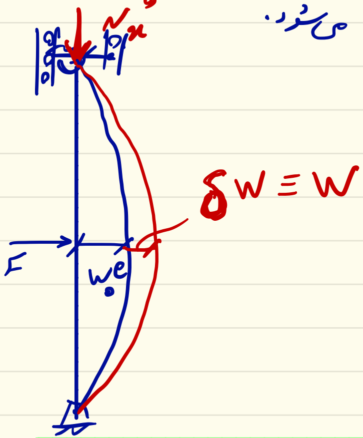
6-2-3 Buckling

کمانش حالت ناپایداری است که اگر کمی جسم تغییر شکل بدهد، دوباره به حالت قبل باز نمی‌گردد و تغییر شکل بی‌نهایت می‌شود.

(6.2-8) $\Rightarrow$

$$\boxed{\frac{d^4W}{dx^4} + \frac{bN_x^0}{E_x^b I_y}\frac{d^2W}{dx^2} = 0} \qquad (6.2\text{-}21)$$

معادله کمانش تیر CLPT



$$w_0 = \underbrace{w_0^e}_{\text{prebuckling}} + \underbrace{W}_{\text{buckling deflection}}$$

معادله (6.2-21) را می‌توان برای جابجایی کل و پیش‌کمانش هم نوشت:

$$E_{xx}^b I_{yy}\frac{d^4w_0}{dx^4} + bN_{xx}(w_0)\frac{d^2w_0}{dx^2} = 0 \qquad (6.2\text{-}22)$$

$$E_{xx}^b I_{yy}\frac{d^4w_0^e}{dx^4} + bN_{xx}(w_0^e)\frac{d^2w_0^e}{dx^2} = 0 \qquad (6.2\text{-}23)$$

با انتگرال گیری از (6.2-21) داریم:

$$\boxed{\frac{d^2 w}{dx^2} + \frac{b N_x^0}{E_x^b I_y} w = k_1 x + k_2} \qquad (6.2\text{-}24)$$

معادله تعادل کمانش تیر CLPT

حل عمومی این معادله چنین است

$$w(x) = C_1 \sin(\lambda_b x) + C_2 \cos(\lambda_b x) + C_3 x + C_4 \qquad (6.2\text{-}25)$$

که در آن

$$\lambda_b^2 = \frac{b N_x^0}{E_x^b I_y} \quad , \quad C_3 = \frac{k_1}{\lambda_b^2} \quad , \quad C_4 = \frac{k_2}{\lambda_b^2} \qquad (6.2\text{-}26)$$

مقادیر $C_1 - C_4$ با استفاده از شرایط مرزی قابل محاسبه است.

ما دنبال آن هستیم که λ_b را به گونه‌ای پیدا کنیم که $w(x)$ غیرصفر بوجود آید.

پس از یافتن λ_b می‌توان نیروی کمانش را یافت.

$$N_x = \left(\frac{E_x^b I_y}{b}\right) \lambda_b^2 \qquad (6.2\text{-}27)$$

که کمترین مقدار آن ، نیروی بحرانی کمانشی می باشد. و از آنجا $w(x)$ شکل مود کمانشی را مشخص می کند.

مثال : تیر دو سر مفصل

$$w_0(0) = 0, \quad w_0(a) = 0, \quad M_{xx}(0) = 0, \quad M_{xx}(a) = 0$$

nditions imply

$$W(0) = 0, \quad W(a) = 0, \quad \frac{d^2W}{dx^2}(0) = 0, \quad \frac{d^2W}{dx^2}(a) = 0$$

$$\begin{aligned}
W(0) = 0: &\quad c_2 + c_4 = 0 \\
W''(0) = 0: &\quad -c_2\lambda_b^2 = 0 \ \text{ which implies } \ c_2 = 0, \ c_4 = 0 \\
W(a) = 0: &\quad c_1 \sin \lambda_b a + c_3 a = 0 \\
W''(a) = 0: &\quad c_1 \sin \lambda_b a = 0 \ \text{ which implies } \ c_3 = 0
\end{aligned}$$

lution, the condition

$$c_1 \sin \lambda_b a = 0 \ \text{ implies that } \ \lambda_b a = n\pi, \ n = 1, 2, \cdots$$

ad is given by

$$bN_{xx}^0 = E_{xx}^b I_{yy} \left(\frac{n\pi}{a}\right)^2$$

e is

$$W(x) = c_1 \sin \frac{n\pi x}{a}, \quad c_1 \neq 0$$

ng load becomes $(n = 1)$

$$N_{cr} = \left(\frac{\pi}{a}\right)^2 \frac{E_{xx}^b I_{yy}}{b} = \left(\frac{\pi^2}{12}\right) \frac{E_{xx}^b h^3}{a^2}$$

node (eigenfunction) associated with it is

$$W(x) = c_1 \sin \frac{\pi x}{a}$$

مثال: تیر دو سر گیردار

$$w_0(0)=0,\quad \frac{dw_0}{dx}(0)=0,\quad w_0(a)=0,\quad \frac{dw_0}{dx}(a)=0$$

ed as

$$W(0)=0,\quad \frac{dW}{dx}(0)=0,\quad W(a)=0,\quad \frac{dW}{dx}(a)=0$$

$$\begin{aligned}
W(0)=0: &\quad c_2+c_4=0\\
W'(0)=0: &\quad c_1\lambda_b+c_3=0\\
W(a)=0: &\quad c_1\sin\lambda_b a+c_2\cos\lambda_b a+c_3 a+c_4=0\\
W'(a)=0: &\quad c_1\lambda_b\cos\lambda_b a-c_2\lambda_b\sin\lambda_b a+c_3=0
\end{aligned}$$

این چهار معادله را می‌توان تبدیل به دو معادله زیر نمود

$$\begin{aligned}
c_1\left(\sin\lambda_b a-\lambda_b a\right)+c_2\left(\cos\lambda_b a-1\right)&=0\\
c_1\left(\cos\lambda_b a-1\right)-c_2\sin\lambda_b a&=0
\end{aligned}$$

برای داشتن جوابهای غیر صفر باید دترمینان ضرایب صفر شود

$$0=\begin{vmatrix}\sin\lambda_b a-\lambda_b a & \cos\lambda_b a-1\\ \cos\lambda_b a-1 & -\sin\lambda_b a\end{vmatrix}$$

$$=\lambda_b a\ \sin\lambda_b a+2\cos\lambda_b a-2$$

به این معادله، معادله مشخصه می‌گویند (*)

characteristic equation

جواب معادله (*) مقادیر ویژه ماتریس ضرایب می باشد.

$$e_n = \lambda_b a$$

یک روش برای حل معادله فوق رسم تابع $f(e_n) = e_n \sin(e_n) + 2\cos(e_n) - 2$ است

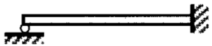
و سپس مشاهده نمود که در چه مقادیری $f(e_n)$ صفر می‌شود. در هر صورت جواب این معادله چنین می‌شود:

$$e_n = 2\pi,\ 8.9868,\ 4\pi,\ 15.45,\ 6\pi,\ \ldots$$

پس نیروی بحرانی کمانش می‌شود.

$$N_{cr} = \left(\frac{e_n}{a}\right)^2 \left(\frac{E^b_{xx} I_{yy}}{b}\right) = \left(\frac{2\pi}{a}\right)^2 \left(\frac{E^b_{xx} I_{yy}}{b}\right)$$
$$= \left(\frac{\pi^2}{3}\right)\left(\frac{E^b_{xx} h^3}{a^2}\right)$$

Table 4.2.2: Values of the constants and eigenvalues for buckling of laminated composite beams with various boundary conditions ($\lambda^2 \equiv bN_{xx}^0/E_{xx}^b I_{yy} = (e_n/a)^2$). The classical laminate theory is used.

End conditions at $x = 0$ and $x = a$	Constants†	Characteristic equation and values* of $e_n \equiv \lambda_n a$
• Hinged-Hinged 	$c_1 \neq 0,\ c_2 = c_3 = c_4 = 0$	$\sin e_n = 0$ $e_n = n\pi$
• Fixed-Fixed 	$c_1 = 1/(\sin e_n - e_n)$ $c_3 = -1/\lambda_n$ $c_2 = -c_4 = 1/(\cos e_n - 1)$	$e_n \sin e_n = 2(1 - \cos e_n)$ $e_n = 2\pi, 8.987, 4\pi, \cdots$
• Fixed-Free 	$c_1 = c_3 = 0$ $c_2 = -c_4 \neq 0$	$\cos e_n = 0$ $e_n = (2n-1)\pi/2$
• Free-Free 	$c_1 = c_3 = 0$ $c_2 \neq 0,\ c_4 \neq 0$	$\sin e_n = 0$ $e_n = n\pi$
• Hinged-Fixed 	$c_1 = 1/e_n \cos e_n,\ c_3 = -1$ $c_2 = c_4 = 0$	$\tan e_n = e_n$ $e_n = 4.493, 7.725, \cdots$

† See Eq. (4.2.28): $W(x) = c_1 \sin \lambda_b x + c_2 \cos \lambda_b x + c_3 x + c_4$.
*For critical buckling load, only the first (minimum) value of $e = \lambda a$ is needed.