جلسه ٢١ مواد مرکب بسم الله الرحمن الرحیم

5-2-5 Laminate Stiffnesses for Selected Laminates

I- Single-Layer Plates

I-a *Single Isotropic Layer*

$$A_{11} = \frac{Eh}{1-\nu^2}\,, \quad A_{12} = \nu A_{11}, \quad A_{22} = A_{11}, \quad A_{66} = \frac{1-\nu}{2}A_{11}, \quad A_{44} = A_{55} = \frac{1-\nu}{2}A_{11}$$

$$D_{11} = \frac{Eh^3}{12(1-\nu^2)}\,, \quad D_{12} = \nu D_{11}, \quad D_{22} = D_{11}, \quad D_{66} = \frac{1-\nu}{2}D_{11} \qquad (3.5.1)$$

(5.2-25)

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & \nu A_{11} & 0 \\ \nu A_{11} & A_{11} & 0 \\ 0 & 0 & \frac{1-\nu}{2}A_{11} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} \text{ (lb/in.)}$$

(5.2-26)

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & \nu D_{11} & 0 \\ \nu D_{11} & D_{11} & 0 \\ 0 & 0 & \frac{1-\nu}{2}D_{11} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} \text{ (lb-in/in.)}$$

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = K\frac{1-\nu}{2}\begin{bmatrix} A_{11} & 0 \\ 0 & A_{11} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \end{Bmatrix} \text{ (lb-in)}$$

$$N_{xx}^T = N_{yy}^T = \frac{E\alpha}{(1-\nu)}\int_{-\frac{h}{2}}^{\frac{h}{2}} \Delta T\, dz, \quad M_{xx}^T = M_{yy}^T = \frac{E\alpha}{(1-\nu)}\int_{-\frac{h}{2}}^{\frac{h}{2}} \Delta T z\, dz$$

(5.2-27)

I-b **Single Specially Orthotropic Layer**

$$A_{11} = Q_{11}h, \quad A_{12} = Q_{12}h, \quad A_{22} = Q_{22}h$$
$$A_{66} = Q_{66}h, \quad A_{44} = Q_{44}h, \quad A_{55} = Q_{55}h \qquad (5.2\text{-}28)$$
$$D_{11} = \frac{Q_{11}h^3}{12}, \quad D_{12} = \frac{Q_{12}h^3}{12}, \quad D_{22} = \frac{Q_{22}h^3}{12}, \quad D_{66} = \frac{Q_{66}h^3}{12}$$

که در آن

$$Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}}, \quad Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}, \quad Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}} \qquad (5.2\text{-}29)$$
$$Q_{66} = G_{12}, \quad Q_{44} = G_{23}, \quad Q_{55} = G_{13}$$

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = h \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} \qquad (5.2\text{-}30)$$

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \frac{h^3}{12} \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix}$$

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = Kh \begin{bmatrix} Q_{44} & 0 \\ 0 & Q_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \end{Bmatrix}$$

$$\begin{Bmatrix} N_{xx}^T \\ N_{yy}^T \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} \int_{-\frac{h}{2}}^{\frac{h}{2}} \Delta T \, dz$$

$$\begin{Bmatrix} M_{xx}^T \\ M_{yy}^T \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} \int_{-\frac{h}{2}}^{\frac{h}{2}} \Delta T z \, dz$$

(5.2-31)

I-C Single Generally Orthotropic Layer

$$A_{ij} = \bar{Q}_{ij} h, \quad D_{ij} = \frac{\bar{Q}_{ij} h^3}{12}, \quad A_{44} = h\bar{Q}_{44}, \; A_{55} = h\bar{Q}_{55}$$

(5.2-32)

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix}$$

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix}$$

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = K \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \end{Bmatrix}$$

(5.2-33)

$$\begin{Bmatrix} N_{xx}^T \\ N_{yy}^T \\ N_{xy}^T \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \alpha_{xx} \\ \alpha_{yy} \\ 2\alpha_{xy} \end{Bmatrix} \int_{-\frac{h}{2}}^{\frac{h}{2}} \Delta T dz$$

(5.2-34)

If the temperature increment is linear through the layer thickness, $\Delta T = T_0 + zT_1$,

$$\begin{Bmatrix} N_{xx}^T \\ N_{yy}^T \\ N_{xy}^T \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \alpha_{xx} \\ \alpha_{yy} \\ 2\alpha_{xy} \end{Bmatrix} T_0 h$$

(5.2-35)

$$\begin{Bmatrix} M_{xx}^T \\ M_{yy}^T \\ M_{xy}^T \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \alpha_{xx} \\ \alpha_{yy} \\ 2\alpha_{xy} \end{Bmatrix} \frac{T_1 h^3}{12}$$

II - **Symmetric Laminates**

II-a *Symmetric Laminates with Multiple Isotropic Layers*

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{11} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix}$$

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{11} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix}$$

(5.2-36)

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = K \begin{bmatrix} A_{44} & 0 \\ 0 & A_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \end{Bmatrix}$$

II-b Symmetric Laminates with Multiple Specially Orthotropic Layers

روابط مانند (5-2-36) می‌باشد.

II-C Symmetric Laminates with Multiple Generally Orthotropic Layers

ماتریس های A و B کاملاً پر هستند

III- **Antisymmetric Laminates**

در حالت کلی

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} \quad (3.$$

(5.2-37)

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} \quad (3.$$

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = K \begin{bmatrix} A_{44} & 0 \\ 0 & A_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \end{Bmatrix} \quad (3$$

III_a Antisymmetric Cross-ply Laminates

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + \begin{bmatrix} B_{11} & 0 & 0 \\ 0 & -B_{11} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix}$$

(5.2-38)

(3.

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & 0 & 0 \\ 0 & -B_{11} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix}$$

(3.

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = K \begin{bmatrix} A_{44} & 0 \\ 0 & A_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \end{Bmatrix}$$

(3

III_b Antisymmetric Angle-ply Laminates

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + \begin{bmatrix} 0 & 0 & B_{16} \\ 0 & 0 & B_{26} \\ B_{16} & B_{26} & 0 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix}$$

(5.2-39)

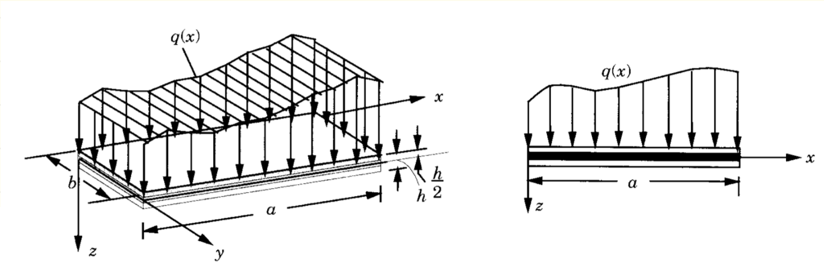
$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & B_{16} \\ 0 & 0 & B_{26} \\ B_{16} & B_{26} & 0 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix}$$

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = K \begin{bmatrix} A_{44} & 0 \\ 0 & A_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \end{Bmatrix}$$

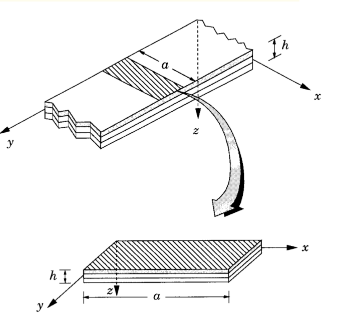
Chapter VI One-Dimensional Analysis of Laminated Composite Plates

6.1 Introduction

- Beam — بعد در جهت y نسبت به x خیلی کم باشد — Plane stress
- cylindrical bending — در جهت y طول بی‌نهایت، مسئله نسبت به y تفاوت ندارد $\left(\frac{\partial}{\partial y}=0\right)$ — Plane strain



Beam



cylindrical

governing equations →

- exact solution: معادله حاکم + شرایط مرزی + شرایط اولیه را کاملا ارضا کند
- numerical solution: " " + " " + " به صورت تقریبی "

FEM, finite difference, boundary element

exact solution

- closed-form: تعداد جملات محدود است ($u(x) = 2 - x^2 + \sin x$ مثلاً)
- infinite series: ($u(x) = \sum_{i=1}^{\infty} a_i \sin(i\pi x)$ مثلاً)

$$u(x) = \sum_{i=1}^{N} a_i \sin(i\pi x)$$

exact - Analytical Solution (aproximate)

6-2 Analysis of Laminated Beam Using CLPT

6-2-1 Governing Equation

فرض می‌کنیم فقط خمش تیر مد نظر باشد. لذا $\{N\} = \{0\}$. پس معادله فضای حالت چنین خواهد شد:

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = -\begin{bmatrix} D \end{bmatrix} \begin{Bmatrix} \frac{\partial^2 w_0}{\partial x^2} \\ \frac{\partial^2 w_0}{\partial y^2} \\ 2\frac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix} \quad (6.2\text{-}1)$$

البته برای خمش تیرها فرض می‌شود

$$M_y = M_{xy} = 0 \quad (6.2\text{-}2)$$

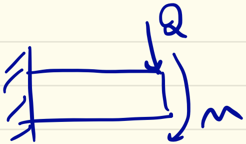
$$\Rightarrow \begin{Bmatrix} \frac{\partial^2 w_0}{\partial x^2} \\ \frac{\partial^2 w_0}{\partial y^2} \\ 2\frac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix} = -\begin{bmatrix} D_{11}^* & D_{12}^* & D_{16}^* \\ D_{21}^* & D_{22}^* & D_{26}^* \\ D_{16}^* & D_{26}^* & D_{66}^* \end{bmatrix} \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} \quad (6.2\text{-}3)$$

اگر تیر به اندازه کافی طویل باشد می‌توان اثر ضریب پواسون و برش را صرف نظر کرد پس

$$w_0 = w_0(x, t) \qquad (6.2\text{-}3)$$

$$\boxed{\frac{\partial^2 w_0}{\partial x^2} = -D_{11}^* M_{xx}} \qquad (6.2\text{-}4)$$

معادله فضای حالت در خمش تیر



در مقاومت مصالح این رابطه را به گونه‌ای دیگر داشتیم. برای رسیدن به آن رابطه

$$M = bM_{xx} \;,\; Q = bQ_x \;,\; E_x^b = \frac{12}{h^3 D_{11}^*} = \frac{b}{I_y D_{11}^*} \quad \text{و} \quad I_y = \frac{bh^3}{12} \qquad (6.2\text{-}5)$$

$$\Rightarrow \frac{\partial^2 w_0}{\partial x^2} = -\frac{M}{E^b I_y} \quad \text{or} \quad M(x) = -EI_y \frac{\partial^2 w_0}{\partial x^2} \qquad (6.2\text{-}6)$$

اما برای یافتن معادله حاکم بر تیر، از معادله حاکم بر ورق CLPT (5.1-21) با قرار دادن $\frac{\partial}{\partial y} = 0$ می‌توان استفاده کرد.

$$\frac{\partial^2 M_{xx}}{\partial x^2} + \hat{N}_x \frac{\partial^2 w_0}{\partial x^2} + q = I_0 \frac{\partial^2 w_0}{\partial t^2} - I_2 \frac{\partial^4 w_0}{\partial x^2 \partial t^2} \qquad (6.2\text{-}7)$$

(فقط معادله سوم)

اگر لمینیت متقارن باشد (آنگاه خمش با کشش مستقل هستند لذا از رابطه 6.2-4 می توان استفاده کرد)

و به اندازه کافی طویل باشد:

$$6.2\text{-}4 \longrightarrow \boxed{-\frac{\partial^2}{\partial x^2}\left(E_x^b I_y \frac{\partial^2 w_0}{\partial x^2}\right) + b\hat{N}_x \frac{\partial^2 w_0}{\partial x^2} + \hat{q} = \hat{I}_0 \frac{\partial^2 w_0}{\partial t^2} - \hat{I}_2 \frac{\partial^4 w_0}{\partial x^2 \partial t^2}} \qquad (6.2\text{-}8)$$

معادله حاکم بر تیر CLPT (طویل و متقارن)

$\hat{N}_x$ نیروی محوری وارد بر تیر است. و

$$\hat{q} = bq, \quad \hat{I}_0 = bI_0, \quad \hat{I}_2 = bI_2, \quad I_i = b\int_{-h/2}^{h/2} \rho z^i \, dz \quad i = 0,1,2 \qquad (6.2\text{-}9)$$

شرایط مرزی

Geometric $\quad w_0, \frac{\partial w_0}{\partial x} \quad$ specify

Force $\quad Q \equiv \frac{\partial M}{\partial x}, M \quad$ specify

6-2-2 Bending

با قرار دادن $\hat{N}_x = 0$ ، معادله (6.2-6) و (6.2-8) چنین خواهند شد:

$$\frac{d^2 w_0}{dx^2} = -\frac{M}{E_x^b I_y} \quad \text{و} \quad E_x^b I_y \frac{d^4 w_0}{dx^4} = \hat{q} \qquad (6.2\text{-}11\ a,b)$$

هر دو معادله فوق، معادلات حاکم هستند. معادله (a) وقتی استفاده می‌شود که ممان خمشی M را برحسب بارواردی بتوان نوشت (یعنی M مشخص باشد) و برای تیرهای نامشخص معادله (b) بیشتر استفاده می‌شود.

General Solution

6.2-11a $\longrightarrow$ $$E_x^b I_y w_0(x) = -\int_0^x \left[\int_0^\eta M(\xi)\, d\xi\right] d\eta + b_1 x + b_2 \qquad (6.2\text{-}12)$$

6.2-11b $\longrightarrow$ $$E_{xx}^b I_{yy} w_0(x) = \int_0^x \left\{\int_0^\xi \left[\int_0^\eta \left(\int_0^\zeta \hat{q}(\mu) d\mu\right) d\zeta\right] d\eta\right\} d\xi + c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3 x + c_4 \qquad (6.2\text{-}13)$$