بسم الله الرحمن الرحیم مکانیک رشد جلسه ۴

problem $\vec{\nabla} \cdot \overleftrightarrow{D} = ?$

$$\vec{\nabla} \cdot \overleftrightarrow{D} = \left(\vec{e}_k \frac{\partial}{\partial y_k}\right) \cdot \left(D_{ij}\, \vec{e}_i \otimes \vec{e}_j\right) = \frac{\partial D_{ij}}{\partial y_k} \overbrace{(\vec{e}_k \cdot \vec{e}_i)}^{\delta_{ki}} \vec{e}_j = \frac{\partial D_{ij}}{\partial y_i} \vec{e}_j$$

(a vector)

Problem: $\vec{A} \otimes \vec{B} \rightsquigarrow$ dyadic

$$\vec{A} \otimes \vec{B} = (A_i \vec{e}_i) \otimes (B_j \vec{e}_j) = A_i B_j (\vec{e}_i \otimes \vec{e}_j) = a_{ki} A'_k \, a_{lj} B'_l (\vec{e}_i \otimes \vec{e}_j)$$

$$= A'_k B'_l \, \vec{e}\,'_k \otimes \vec{e}\,'_l$$

$$\vec{F} = F_i \vec{e}_i = F'_i \vec{e}\,'_i$$

$$\vec{F} = m \vec{a} \qquad F_i = m a_i$$

$$F'_i = m a'_i$$

تعریف: Double-dot product

$$\boxed{(\vec{A}\otimes\vec{B}):(\vec{C}\otimes\vec{D}) = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D})}$$

problem $\overleftrightarrow{\Phi} : \overleftrightarrow{\Psi} = ?$

$$= \Phi_{ij}(\vec{e}_i\otimes\vec{e}_j) : \Psi_{kl}(\vec{e}_k\otimes\vec{e}_l) = \Phi_{ij}\Psi_{kl}(\vec{e}_i\otimes\vec{e}_j):(\vec{e}_k\otimes\vec{e}_l)$$

$$= \Phi_{ij}\Psi_{kl}\underbrace{(\vec{e}_i\cdot\vec{e}_k)}_{\delta_{ik}}\underbrace{(\vec{e}_j\cdot\vec{e}_l)}_{\delta_{jl}} = \Phi_{ij}\Psi_{ij}$$

$$\boxed{\overleftrightarrow{\Phi} : \overleftrightarrow{\Psi} = \Phi_{ij}\Psi_{ij}}$$

problem $\overleftrightarrow{\Phi} : \overleftrightarrow{I} = ?$

$$= \dots = \Phi_{ii}$$

تمرین سری ١ :

1- $\vec{u}\cdot(\vec{v}\times\vec{w})=\vec{v}\cdot(\vec{w}\times\vec{u})=\vec{w}\cdot(\vec{u}\times\vec{v})$

2- $\vec{u}\cdot(\vec{u}\times\vec{v})=0$

3- $\vec{u}\times\vec{v}=\det\begin{bmatrix} e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$

4- $(\overleftrightarrow{A}\cdot\overleftrightarrow{B})^T=B^T\cdot A^T$

5- $I:\overleftrightarrow{A}=tr(\overleftrightarrow{A})=A:I$

6- $\overleftrightarrow{A}:(\overleftrightarrow{B}\cdot\overleftrightarrow{C})=(\overleftrightarrow{B}^T\cdot\overleftrightarrow{A}):\overleftrightarrow{C}=(\overleftrightarrow{A}\cdot\overleftrightarrow{C}^T):\overleftrightarrow{B}$

7- $(\overleftrightarrow{A}\cdot\overleftrightarrow{B})^{-1}=\overleftrightarrow{B}^{-1}\cdot\overleftrightarrow{A}^{-1}$

8- $\overleftrightarrow{S}:\overleftrightarrow{B}=\overleftrightarrow{S}:\overleftrightarrow{B}^T=\overleftrightarrow{S}:\frac{1}{2}(\overleftrightarrow{B}+\overleftrightarrow{B}^T)$ (S: Symetric tensor)

9- $\overleftrightarrow{S}:\overleftrightarrow{W}=\overleftrightarrow{0}$ (W: Skew (Antisymetric tensor)

10- $\overleftrightarrow{A}^{-T}:\overleftrightarrow{A}=3$

problem $\vec{e}_i \cdot \overleftrightarrow{A} \cdot \vec{e}_j = ?$

$$= \vec{e}_i \cdot (A_{kl}\, \vec{e}_k \otimes \vec{e}_l) \cdot \vec{e}_j = A_{kl}\,\delta_{ik}\,\delta_{lj}$$

$$= A_{ij}$$

تعریف: Dot production a dyadic and a dyadic

$$(\vec{u} \otimes \vec{v}) \cdot (\vec{w} \otimes \vec{x}) = (\vec{v} \cdot \vec{w})\,(\vec{u} \otimes \vec{x})$$

a dyadic

problem $\overleftrightarrow{A} \cdot \overleftrightarrow{B} = ?$

$$(A_{ij}\, \vec{e}_i \otimes \vec{e}_j) \cdot (B_{kl}\, \vec{e}_k \otimes \vec{e}_l)$$

$$= A_{ij} B_{kl}\, \underbrace{(\vec{e}_j \cdot \vec{e}_k)}_{\delta_{jk}}\, (\vec{e}_i \otimes \vec{e}_l)$$

$$= A_{ij} B_{jl}\, \vec{e}_i \otimes \vec{e}_l$$

به عبارت دیگر $\boxed{(A\cdot B)_{ij} = A_{ik} B_{kj}}$ $A, B$: dyadic

یا $\vec{e}_i \cdot (A \cdot B) \cdot \vec{e}_j = \dots = A_{ik} B_{kj}$

useful properties:

vector: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

$$(\alpha+\beta)\vec{u} = \alpha\vec{u} + \beta\vec{u}$$

$$\alpha(\vec{u}+\vec{v}) = \alpha\vec{u} + \alpha\vec{v}$$

$$\vec{u}\cdot(\alpha\vec{v}+\beta\vec{w}) = \alpha(\vec{u}\cdot\vec{v}) + \beta(\vec{u}\cdot\vec{w})$$

$$|\vec{u}| = \sqrt{\vec{u}\cdot\vec{u}} = \sqrt{u_i u_i}$$

$2^{nd}$-order tensors and dyadics :

$$\overleftrightarrow{A}\cdot(\alpha\vec{u}+\vec{v}) = \alpha\,\overleftrightarrow{A}\cdot\vec{u} + \overleftrightarrow{A}\cdot\vec{v}$$

$$(\overleftrightarrow{A} \pm \overleftrightarrow{B})\cdot\vec{u} = \overleftrightarrow{A}\cdot\vec{u} \pm \overleftrightarrow{B}\cdot\vec{u}$$

$$(\alpha\overleftrightarrow{A})\cdot\vec{u} = \alpha(\overleftrightarrow{A}\cdot\vec{v})$$

$$(\alpha\vec{u}+\beta\vec{v})\otimes\vec{w} = \alpha(\vec{u}\otimes\vec{w}) + \beta(\vec{v}\otimes\vec{w})$$

Dyadics:

$A, B, C$: dyadics

$$A+B = B+A$$

$$(A\cdot B)\cdot C = A\cdot(B\cdot C) = A\cdot B\cdot C$$

$$A^2 = A\cdot A$$

$$(A+B)\cdot C = A\cdot C + B\cdot C$$

تعریف : Invers of a dyadic

$$A\cdot A^{-1} = I = A^{-1}\cdot A$$

useful properties:

A: Dyadic

$$(A^{-1})^{-1} = A$$

$$(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$$

$$(A^{-1})^T = (A^T)^{-1}$$

$$\det(A^{-1}) = (\det(A))^{-1}$$

$$(A.B\ldots y.z)^{-1} = z^{-1}.y^{-1}\ldots B^{-1}.A^{-1}$$

$$(A.B\ldots y.z)^T = z^T.y^T\ldots B^T.A^T$$

## Symetric and skew tensors

A : second-order tensor

$$S = \frac{1}{2}(A + A^T) \quad , \quad W = \frac{1}{2}(A - A^T)$$

symetric tensor      skew tensor

useful properties:

B: dyadic

$$S:B = S:B^T = S:\tfrac{1}{2}(B+B^T)$$

$$W:B = -W:B^T = W:\tfrac{1}{2}(B-B^T)$$

$$S:W = 0$$

تعریف adjoint of a dyadic

$$\mathrm{adj}(\overleftrightarrow{A}) = A^{adj} = \det(A)\,A^{-1} \qquad \left(\equiv \frac{\det(A)}{A}\right)$$

تعریف cofactor of a dyadic

$$\mathrm{cof}(\overleftrightarrow{A}) = A^{cof} = \det(A)\,A^{-T} = \left(A^{adj}\right)^T$$

$$\left(\equiv \frac{\det(A)}{A^t}\right)$$

# 1.9- Higher order Tensors

$$A_{ij\cdots n}\,\vec{e}_i\otimes\vec{e}_j\cdots\otimes\vec{e}_n$$

تعريف $(\vec{u}\otimes\vec{v}\otimes\vec{w})\cdot\vec{x}=(\vec{w}\cdot\vec{x})(\vec{u}\otimes\vec{v})$

تعريف $(\vec{u}\otimes\vec{v}\otimes\vec{w}):(\vec{x}\otimes\vec{y})=(\vec{v}\cdot\vec{x})(\vec{w}\cdot\vec{y})\,\vec{u}$

problem $(\vec{u}\otimes\vec{v}\otimes\vec{w}):\underset{\sim}{I}=?$ $\quad \underset{\sim}{I} = (\vec{e}_i\otimes\vec{e}_i)$

$$=\cdots=(\vec{v}\cdot\vec{w})\,\vec{u}$$

تعريف $(\vec{u}\otimes\vec{v}\otimes\vec{w}\otimes\vec{x}):(\vec{y}\otimes\vec{z})=(\vec{w}\cdot\vec{y})(\vec{x}\cdot\vec{z})(\vec{u}\otimes\vec{v})$

problem: $\quad \mathbb{A}$ : a tensor of order four

$\qquad\qquad\;\; B$ : a second-order tensor

$$\mathbb{A} : B = A_{ijkl} \, B_{kl} \, \vec{e}_i \otimes \vec{e}_j$$

useful properties:

$A, B, C, D$ : $2^{nd}$-order tensors

$$(A \otimes B) : C = A \, (B : C) = (B : C) \, A$$

$$A : (B \otimes C) = (A : B) \, C = C \, (A : B)$$

$$(A \otimes B) : (C \otimes D) = (B : C) \, (A \otimes D) = (A \otimes D)(B : C)$$

$$A \cdot (B \otimes C) \cdot D = (A \cdot B) \otimes (C \cdot D)$$

تعریف $\quad \mathbb{I}, \bar{\mathbb{I}}$ : fourth-order unit tensor

$$\overleftrightarrow{A} = \mathbb{I} : \overleftrightarrow{A} \quad , \quad \overleftrightarrow{A}^T = \bar{\mathbb{I}} : \overleftrightarrow{A}$$

$$(\mathbb{I})_{ijkl} = \delta_{ik}\,\delta_{jl} \longrightarrow \mathbb{I} = \delta_{ik}\,\delta_{jl}\,\vec{e}_i \otimes \vec{e}_j \otimes \vec{e}_k \otimes \vec{e}_l = \vec{e}_i \otimes \vec{e}_j \otimes \vec{e}_i \otimes \vec{e}_j$$

$$(\bar{\mathbb{I}})_{ijkl} = \delta_{il}\,\delta_{jk} \longrightarrow \bar{\mathbb{I}} = \delta_{il}\,\delta_{jk}\,\vec{e}_i \otimes \vec{e}_j \otimes \vec{e}_k \otimes \vec{e}_l = \vec{e}_i \otimes \vec{e}_j \otimes \vec{e}_j \otimes \vec{e}_i$$

تعریف $\quad (\mathbb{A}^T)_{ijkl} = \mathbb{A}_{klij}$

$A, B =$ 2nd order $\quad (A \otimes B)^T = B \otimes A$

## تمرین سری ۲ :

Second-order tensor : D, C, B, A, S

First-order tensor : u, v

1- $\vec{u}\cdot(\overleftrightarrow{S}\cdot\vec{v}) = \vec{v}\cdot(\overleftrightarrow{S}^t\cdot\vec{u})$

2- $(\vec{u}\otimes\vec{v})\cdot\overleftrightarrow{S} = \vec{u}\otimes\overleftrightarrow{S}^t\cdot\vec{v}$

3- $tr(\vec{u}\otimes\vec{v}) = (\vec{u}\otimes\vec{v})_{ii} = u_i v_i$

4- $tr(S^t) = tr(S)$

5- $tr(S_1S_2) = tr(S_2S_1)$

6- $A:B = tr(A^t\cdot B) = tr(B\cdot A^t) = tr(B^t\cdot A) = tr(A\cdot B^t) = A_{ij}B_{ij}$

7- $tr(S) = I:S$

8- $tr(I) = 3$

9- $C\cdot(A\otimes B)\cdot D = (C\cdot A)\otimes(B\cdot D)$

10- $S\cdot(\vec{u}\otimes\vec{v}) = S\cdot\vec{u}\otimes\vec{v}$

11- $(\vec{u}\otimes\vec{v})\cdot S = \vec{u}\otimes S^t\cdot\vec{v}$