بسم الله الرحمن الرحیم — مکانیک رشد — جلسه ۸

Example $\dfrac{\partial A^2}{\partial A} = ?$

$$\left(\frac{\partial A^2}{\partial A}\right)_{ijkl} = \frac{\partial (A\cdot A)_{ij}}{\partial A_{kl}} = \frac{\partial (A_{im}A_{mj})}{\partial A_{kl}} = \frac{\partial A_{im}}{\partial A_{kl}} A_{mj} + A_{im}\frac{\partial A_{mj}}{\partial A_{kl}}$$

$$= \delta_{ik}\delta_{ml}A_{mj} + A_{im}\delta_{mk}\delta_{jl} = \delta_{ik}A_{lj} + A_{ik}\delta_{jl}$$

Problem $\dfrac{\partial I_1}{\partial A} = ?$

$$I_1 = tr(A) = A_{ii} \Rightarrow \left(\frac{\partial I_1}{\partial A}\right)_{kl} = \frac{\partial A_{ii}}{\partial A_{kl}} = \delta_{ik}\delta_{il} = \delta_{kl}$$

$$\Rightarrow \frac{\partial I_1}{\partial A} = I$$

problem $\frac{\partial I_2}{\partial A} = ?$

$$I_2 = \frac{1}{2}\left(tr(A)\right)^2 - \frac{1}{2} tr(A^2)$$

$$\frac{\partial I_2}{\partial A} = \frac{1}{2} \times 2\, tr(A) \frac{\partial tr(A)}{\partial A} - \frac{1}{2} \frac{\partial tr(A^2)}{\partial A} = tr(A)\, I - \frac{1}{2} \frac{\partial tr(A^2)}{\partial A}$$

$$\left(\frac{\partial tr(A^2)}{\partial A}\right)_{kl} = \frac{\partial (A_{im} A_{mi})}{\partial A_{kl}} = \delta_{ik} A_{li} + A_{ik}\delta_{il} = A_{lk} + A_{lk} = 2A_{lk}$$

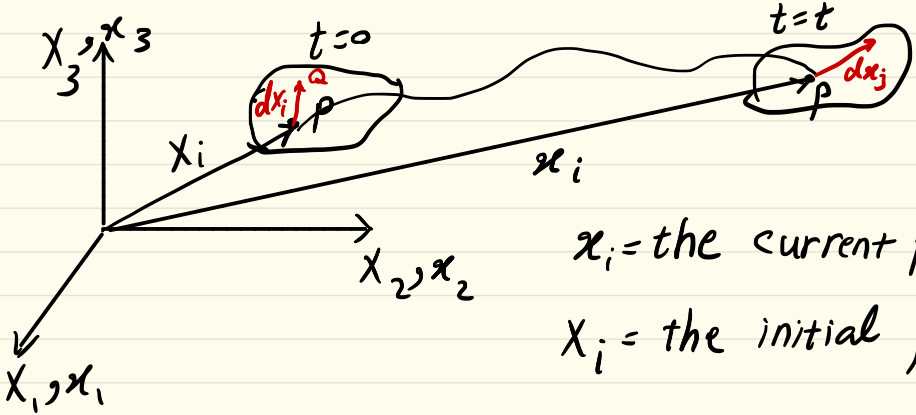
$$\Rightarrow \frac{\partial tr(A^2)}{\partial A} = 2A^T$$

$$\Rightarrow \frac{\partial I_2}{\partial A} = tr(A)\, I - \frac{1}{2} \times 2A^T = tr(A)\, I - A^T$$

problem $\frac{\partial I_3}{\partial A} = ?$

$$I_3 = \det(A) \Rightarrow \frac{\partial \det(A)}{\partial A} = \det(A)\, A^{-T}$$

Chapter 2 - Analysis of Kinematic problem



$x_i =$ the current position of point "P"

$X_i =$ the initial position of point "P"

$$x_i = x_i(X_j, t) = x_i(X, t) \quad , \quad X_i = X_i(x_j, t) = X_i(x, t)$$

$$\left.\begin{aligned}
dx_1 &= \frac{\partial x_1}{\partial X_1} dX_1 + \frac{\partial x_1}{\partial X_2} dX_2 + \frac{\partial x_1}{\partial X_3} dX_3 \\
dx_2 &= \frac{\partial x_2}{\partial X_1} dX_1 + \frac{\partial x_2}{\partial X_2} dX_2 + \frac{\partial x_2}{\partial X_3} dX_3 \\
dx_3 &= \frac{\partial x_3}{\partial X_1} dX_1 + \frac{\partial x_3}{\partial X_2} dX_2 + \frac{\partial x_3}{\partial X_3} dX_3
\end{aligned}\right\} \Rightarrow \begin{Bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{Bmatrix} = \overbrace{\begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \end{bmatrix}}^{\text{تانسور گرادیان تغییر شکل } F} \begin{Bmatrix} dX_1 \\ dX_2 \\ dX_3 \end{Bmatrix}$$

تعریف:

$$\boxed{F_{ij} = \frac{\partial x_i}{\partial X_j}}$$

تانسور گرادیان تغییر شکل

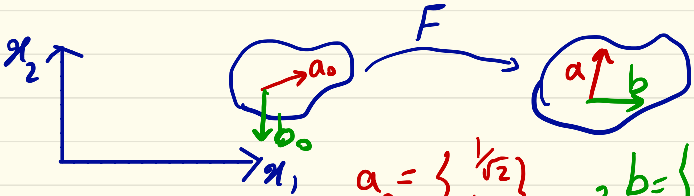
$$d\vec{x} = \vec{F} \cdot d\vec{X} \quad , \quad \{dx\} = [F]\{dX\}$$

تعریف

$$\boxed{J = |F|}$$

ژاکوبین

Example



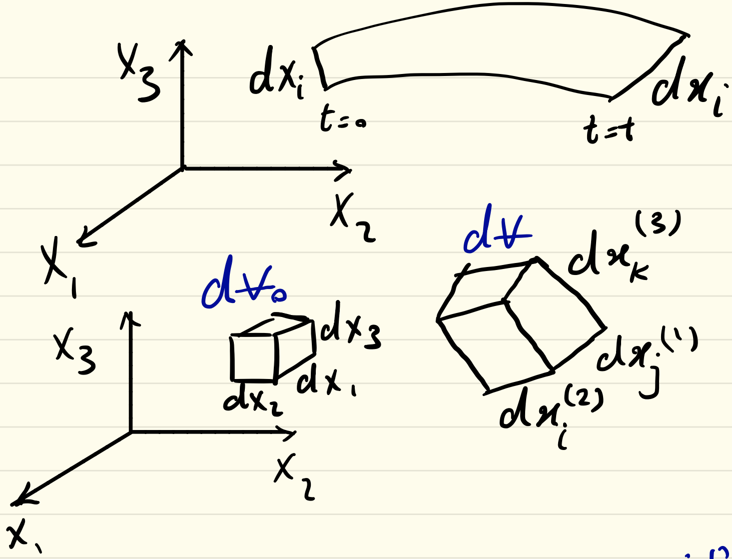
$$\begin{cases} x_1 = 4 - 2X_1 - X_2 \\ x_2 = 2 + \frac{3}{2}X_1 - \frac{1}{2}X_2 \end{cases}$$

$$a_0 = \begin{Bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{Bmatrix} \quad , \quad b = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

$$F = \begin{bmatrix} -2 & -1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \Rightarrow [F]^{-1} = \begin{bmatrix} -\frac{1}{5} & \frac{2}{5} \\ -\frac{3}{5} & -\frac{4}{5} \end{bmatrix}$$

$$\{a\} = [F] \cdot \{a_0\} = \begin{bmatrix} -2 & -1 \\ 3/2 & -1/2 \end{bmatrix} \begin{Bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{Bmatrix} = \frac{1}{\sqrt{2}} \begin{Bmatrix} -3 \\ 1 \end{Bmatrix}$$

$$\{b_0\} = [F]^{-1} \cdot \{b\} = \begin{bmatrix} -1/5 & 2/5 \\ -3/5 & -4/5 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = -\frac{1}{5} \begin{Bmatrix} 1 \\ 3 \end{Bmatrix}$$



$$dx_i = \frac{\partial x_i}{\partial X_j} dX_j$$

حجم $dV = (dx_i^{(1)} \vec{e}_i) \cdot (dx_j^{(2)} \vec{e}_j \times dx_k^{(3)} \vec{e}_k)$

$$\boxed{dV = J\, dV_0}$$

حجم ثانویه

حجم اولیه

تعریف : اگر در یک تغییر شکلی، F به این صورت باشد

$J = |F| = 1 \Rightarrow$ F isocharic or Volume-Preserive

(حجم ثابت)

Nanson formula

$$dV = J\, dV_0 \longrightarrow \vec{da}\cdot\vec{dx} = J\,\vec{dA}\cdot\vec{dX}$$

$$\vec{da}\cdot F\cdot\vec{dX} = J\,\vec{dA}\cdot\vec{dX}$$

می دانیم $\vec{u}\cdot\overleftrightarrow{A} = \overleftrightarrow{A}^{t}\cdot\vec{u}$

$$F^{T}\cdot\vec{da} = J\,\vec{dA} \Rightarrow \boxed{\vec{da} = J\,F^{-T}\cdot\vec{dA}}$$

رابطه نانسون

$$da\,\vec{n} = J\,F^{-T}\cdot dA\,\vec{n}_0 \Rightarrow \frac{da}{dA} = \left|J\,F^{-T}\,\vec{n}_0\right|$$

بردار یکه

تعریف

$$\boxed{\eta = \frac{da}{dA} = \left|J\,F^{-T}\cdot\vec{n}_0\right| \quad \text{area change}}$$

بصورت مشابه

$$\vec{dx} = F\cdot\vec{dX} \Rightarrow dx\,\vec{n} = F\cdot dX\,\vec{n}_0$$

$$\Rightarrow \quad \frac{dx}{dX} = |F \cdot \vec{n}_0|$$

تعریف :

$$\boxed{\lambda = \frac{dx}{dX} = |F \cdot \vec{n}_0| \qquad \text{Stretch}}$$