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مكانيك رشد جلسه ۳

1.5 - Gradient, Divergence, curl

$\phi = \phi(y_1, y_2, y_3) \quad (\equiv \phi(y))$ a scaler field

$\vec{A} = \vec{A}(y_1, y_2, y_3)$ a vector field

$\vec{\nabla}$: del operator (Nabla) $= \vec{e}_1 \frac{\partial}{\partial y_1} + \vec{e}_2 \frac{\partial}{\partial y_2} + \vec{e}_3 \frac{\partial}{\partial y_3}$

$$\boxed{\vec{\nabla} = \vec{e}_i \frac{\partial}{\partial y_i}}$$

gradient of ϕ: $\quad \vec{\nabla}\phi = \vec{e}_1 \frac{\partial \phi}{\partial y_1} + \vec{e}_2 \frac{\partial \phi}{\partial y_2} + \vec{e}_3 \frac{\partial \phi}{\partial y_3}$

$$\boxed{\vec{\nabla}\phi = \vec{e}_i \frac{\partial \phi}{\partial y_i} \equiv \vec{e}_i \phi_{,i}}$$

problem: prove that del operator is an invarient. $(B_{ijk} = B'_{ijk})$

$$\vec{\nabla} = \vec{e}_i \frac{\partial}{\partial y_i} = \vec{e}'_i \frac{\partial}{\partial y'_i}$$

$$\vec{e}_i = a_{ji} \vec{e}'_j \qquad \frac{\partial}{\partial y_i} = \left(\frac{\partial}{\partial y'_i}\right)\left(\frac{\partial y'_i}{\partial y_i}\right)$$

$$y'_k = a_{km} y_m \Longrightarrow \frac{\partial y'_k}{\partial y_i} = a_{km} \underbrace{\frac{\partial y_m}{\partial y_i}}_{\delta_{mi}} = a_{ki} \Rightarrow \frac{\partial}{\partial y_i} = \left(\frac{\partial}{\partial y'_i}\right) a_{ki}$$

$$\Rightarrow \vec{\nabla} = \vec{e}_i \frac{\partial}{\partial y_i} = \underbrace{a_{ji}\ a_{ki}}_{\delta_{jk}} \vec{e}'_j \frac{\partial}{\partial y'_k} = \vec{e}'_j \frac{\partial}{\partial y'_j}$$

تعريف: Laplacian operator ∇^2

$$\nabla^2 = \vec{\nabla} \cdot \vec{\nabla} = \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} + \frac{\partial^2}{\partial y_3^2} = (\quad)_{,ii}$$

$$\nabla^2 \Phi = \Phi_{,ii}$$

Divergence of a vector field:

$$\operatorname{div} \vec{A} = \vec{\nabla} \cdot \vec{A} = \vec{e}_i \frac{\partial}{\partial y_i} \cdot A_j \vec{e}_j = \frac{\partial A_j}{\partial y_i} \underbrace{(\vec{e}_i \cdot \vec{e}_j)}_{\delta_{ij}} = \frac{\partial A_i}{\partial y_i} = A_{i,i}$$

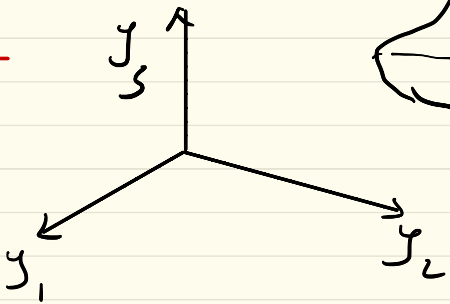
Curl of a vector field:

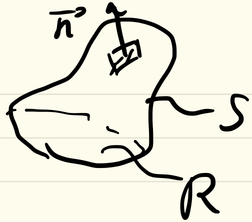
$$\operatorname{Curl}(\vec{A}) = \vec{\nabla} \times \vec{A} = \left(\vec{e}_i \frac{\partial}{\partial y_i}\right) \times \left(A_j \vec{e}_j\right) = \frac{\partial A_j}{\partial y_i} \left(\vec{e}_i \times \vec{e}_j\right)$$

$$= \boxed{\frac{\partial A_j}{\partial y_i} \varepsilon_{ijk} \vec{e}_k}$$

1.6 The Gredient Theorem

$$\Phi = \Phi(y)$$





$$\boxed{\iiint_V \vec{\nabla}\Phi \, dv = \iint_S \vec{n}\,\Phi \, ds}$$

$$\int_R \Phi_{,i} \, dv = \int_S n_i \Phi \, ds$$

1.7 The Divergence Theorem of Gauss:

$\vec{A}$: vector field

$$\boxed{\int_R \vec{\nabla}\cdot\vec{A} \, dv = \int_S \vec{A}\cdot\vec{n} \, ds}$$

$$\int_R A_{i,i} \, dv = \int_S A_i n_i \, ds$$

$$\int_R A_{ij\ldots,r}\,dv = \int_S A_{ij\ldots}\,n_r\,ds$$

1.8- Dyads and Dyadics

$$\vec{F} \equiv F_i \equiv (F_1, F_2, F_3)$$

$D_{ij} = ?$

$\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$

$\vec{e}_i \times \vec{e}_j = \varepsilon_{ijk}\vec{e}_k$

$\otimes \quad \underline{\otimes} \quad \boxtimes$

$\vec{e}_i \otimes \vec{e}_j$ unit dyad

$(\vec{e}_i\vec{e}_j)$

$$\underbrace{\overleftrightarrow{\Phi}}_{\text{dyadic}} = \Phi_{ij}\,(\underbrace{\vec{e}_i \otimes \vec{e}_j}_{\text{dyads}})$$

$[\Phi]$

$$\vec{F} = F_1\vec{e}_1 + F_2\vec{e}_2 + F_3\vec{e}_3$$

$$\overleftrightarrow{\Phi} = \Phi_{11}\,\vec{e}_1\otimes\vec{e}_1 + \Phi_{12}\,\vec{e}_1\otimes\vec{e}_2 + \Phi_{13}\,\vec{e}_1\otimes\vec{e}_3 + \Phi_{21}\,\vec{e}_2\otimes\vec{e}_1 \ldots$$

$$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \rightsquigarrow a\begin{bmatrix}1\\0\end{bmatrix}+b\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}a\\b\end{bmatrix}$$

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \rightsquigarrow \begin{bmatrix}a\\b\\c\end{bmatrix}$$

$$\phi^{(3)} \equiv \overleftrightarrow{\phi} = \phi_{ijk}\,\vec{e}_i \otimes \vec{e}_j \otimes \vec{e}_k$$

problem $\overleftrightarrow{\phi}$: invarient

$$\overleftrightarrow{\phi} = \phi_{ij}\vec{e}_i \otimes \vec{e}_j = a_{ki}a_{lj}\,\phi''_{kl}\,\vec{e}_i \otimes \vec{e}_j = \phi'_{kl}\,\vec{e}\,'_k \otimes \vec{e}\,'_l$$

تعریف: $\overleftrightarrow{I}$ unite dyadic

$$\overleftrightarrow{I} = \vec{e}_1\vec{e}_1 + \vec{e}_2\vec{e}_2 + \vec{e}_3\vec{e}_3 \equiv \vec{e}_i \otimes \vec{e}_i$$

تعریف: dot production a vector $\vec{v}$ and a dyadic $\overleftrightarrow{D}$

$$\vec{v}\cdot\overleftrightarrow{D} = (v_i \vec{e}_i)\cdot(D_{kl}\,\vec{e}_k\otimes\vec{e}_l) = v_i D_{kl}\,\vec{e}_i\cdot(\vec{e}_k\otimes\vec{e}_l)$$

$$= v_i D_{kl}\underbrace{(\vec{e}_i\cdot\vec{e}_k)}_{\delta_{ik}}\vec{e}_l = v_k D_{kl}\,\vec{e}_l$$

تعریف: dot production a dyadic $\overleftrightarrow{D}$ and a vector $\vec{V}$

$$\overleftrightarrow{D}\cdot\vec{V} = (D_{ij}\,\vec{e}_i\otimes\vec{e}_j)\cdot(V_k\,\vec{e}_k) = D_{ij}V_k\,(\vec{e}_i\otimes\vec{e}_j)\cdot\vec{e}_k$$

$$= D_{ij}V_k\,\vec{e}_i\,(\vec{e}_j\cdot\vec{e}_k) = D_{ik}V_k\,\vec{e}_i$$

$$\boxed{\overleftrightarrow{D}\cdot\vec{V} \neq \vec{V}\cdot\overleftrightarrow{D} \quad \text{unless} \quad D_{ij}=D_{ji}}$$

تعریف: cross-produc of a vector $\vec{v}$ with a dyadic $\overleftrightarrow{D}$

$$\vec{v} \times \overleftrightarrow{D} = (v_i \vec{e}_i) \times (D_{kl} \vec{e}_k \otimes \vec{e}_l) = v_i D_{kl} \vec{e}_i \times (\vec{e}_k \otimes \vec{e}_l)$$

$$= v_i D_{kl} \underbrace{(\vec{e}_i \times \vec{e}_k)}_{\varepsilon_{ikj} \vec{e}_j} \otimes \vec{e}_l = v_i D_{kl} \varepsilon_{ikj} (\vec{e}_j \otimes \vec{e}_l)$$

problem: $\overleftrightarrow{D} \cdot \overleftrightarrow{I} = \overleftrightarrow{I} \cdot \overleftrightarrow{D} = \overleftrightarrow{D}$

$$\overleftrightarrow{D} \cdot \overleftrightarrow{I} = (D_{ij} \vec{e}_i \otimes \vec{e}_j) \cdot (\vec{e}_k \otimes \vec{e}_k) = D_{ij} \overbrace{(\vec{e}_j \cdot \vec{e}_k)}^{\delta_{jk}} (\vec{e}_i \otimes \vec{e}_k)$$

$$= D_{ij} \vec{e}_i \otimes \vec{e}_j = \overleftrightarrow{D}$$

تعریف: conjucation or transpose of a dyadic

$$\overleftrightarrow{D}^t = D_{ij} \vec{e}_j \otimes \vec{e}_i = D_{ji} \vec{e}_i \otimes \vec{e}_j$$

تعريف : The scaler of a dyadic $\overleftrightarrow{D}_s$

$$\overleftrightarrow{D}_s = D_{ij} \underbrace{\vec{e}_i \cdot \vec{e}_j}_{\delta_{ij}} = D_{ii} \equiv tr(D_{ij}) = D_{ii}$$

invarient

$$tr(D_{ij}) = tr(D'_{ij})$$

تعريف The vector of a dyadic $\vec{D}_v$

$$\vec{D}_v = D_{ij} \vec{e}_i \times \vec{e}_j = D_{ij} \varepsilon_{ijk} \vec{e}_k$$

تعريف : $\vec{\nabla} \otimes \vec{A} = ?$

$$\vec{\nabla} \otimes \vec{A} = \left(\vec{e}_k \frac{\partial}{\partial y_k}\right) \otimes \left(A_j \vec{e}_j\right) = \vec{e}_k \frac{\partial A_j}{\partial y_k} \otimes \vec{e}_j$$

$$= \frac{\partial A_j}{\partial y_k} \left(\vec{e}_k \otimes \vec{e}_j\right) \quad \text{(a dyadic)}$$